Notation

$X_n \xrightarrow{\mathcal{D}} X$ denotes convergence in distribution. $N(\mu, \sigma)$ denotes normal distribution with mean μ and variance σ^2.

Questions

1. Let $U_i, i = 1, 2, \dots$ be independent random variables, uniformly distributed over $(0, 1)$.

   (a) Set $M_n = \min_{i=1,\dots,n} U_i$. Find $a_n \in \mathbb{R}$ such that $a_n M_n$ converges in distribution to a non-zero random variable.

   (b) Let V_n denote the second smallest random variable among $U_1, \dots, U_n$. Provide an expression for $P(V_n > x)$.

   (c) Using the sequence a_n that you found in part (a), show that $a_n V_n$ converges in distribution.

   Hint: The events $M_n > x$ and $V_n > x$ can be expressed in terms of the binomial random variable $N := \#\{k \le n : U_k > x\}$.

2. Suppose $X_1, X_2, \dots$ are independent identically distributed random variables. Prove that the following conditions are equivalent:

   (a) $X_n/n \to 0$ almost surely.

   (b) $E|X_1| < \infty$.

3. Suppose that $X_1, X_2, \dots$ are independent identically distributed random variables with $E(X_1) = 0$ and $E(X_1^4) < \infty$. Let $S_n = X_1 + \cdots + X_n$. If $\theta > 3/4$, prove that $\frac{1}{n^\theta} S_n \to 0$ with probability one.

4. Suppose that $X_1, X_2, \dots$ are independent random variables with distributions

   $$P(X_k = \pm 1) = \frac{1}{2k} \text{ and } P(X_k = 0) = \frac{1-k}{k} \text{ for } k = 1, 2, \dots.$$

   Prove that

   $$\frac{1}{\sqrt{\ln n}} \sum_{k=1}^{n} X_k \xrightarrow{\mathcal{D}} N(0, 1).$$