REAL ANALYSIS PRELIMINARY EXAM
AUGUST 2020

Time: 2 hours 30 minutes.
Answer all problems and fully justify your work.

Question 1

(1) Suppose that $A, B \subset \mathbb{R}$ are bounded (not necessarily Lebesgue measurable) sets for which there exists $\alpha > 0$ such that $|a - b| \geq \alpha$ for every $a \in A$ and $b \in B$. Prove that

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

(2) Let $E_n \subset \mathbb{R}$ be Lebesgue measurable sets for $n \in \mathbb{N}$. Show that

$$E := \{x \in \mathbb{R} : x \text{ belongs to exactly two of the sets } E_n\}$$

is also Lebesgue measurable.
Hint: One strategy is to use characteristic functions.

(3) State Egorov's Theorem without proof.

(4) Define the sequence $f_n \colon [0,1] \to \mathbb{R}$ by $f_n(x) = n$ for $0 \leq x \leq 1/n$ and $f_n(x) = 0$ for $1/n < x \leq 1$. Identify a function $f \colon [0,1] \to \mathbb{R}$ such that $f_n \to f$ Lebesgue almost everywhere and prove Egorov's theorem in this special case.

Question 2

(1) State without proof the dominated convergence theorem.

(2) Prove that

$$\lim_{n\to\infty} \int_0^\infty \frac{4t^3 + 12}{12t^6 + 3nt + 2} dt = 0.$$

Please turn over for remaining questions.

Question 3

(1) Carefully state Fubini's theorem and Tonelli's theorem.
(2) Fix an integrable function $f\colon \mathbb{R} \to \mathbb{R}$. Define a function $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = \int_{-\infty}^{x} f(t)dt$. Show that if $c \in \mathbb{R}$ is fixed then $g(x+c) - g(x)$ is an integrable function of x and
$$\int_{-\infty}^{\infty} (g(x+c) - g(x))dx = c \int_{-\infty}^{\infty} f(t)dt.$$
(3) Given a set $A \subset \mathbb{R}^2$, recall the vertical and horizontal sections defined for each $x, y \in \mathbb{R}$ by
$$A_x = \{y \in \mathbb{R}\colon (x, y) \in A\}.$$
$$A^y = \{x \in \mathbb{R}\colon (x, y) \in A\}.$$
Decide whether the following statements are true or false for an arbitrary set $A \subset \mathbb{R}^2$.
  (a) If $m(A_x) = 0$ for Lebesgue almost every x then $m(A^y) = 0$ for Lebesgue almost every y.
  (b) If $m(A_x) = 0$ for all x then $m(A^y) = 0$ for all y.

Question 4

(1) Define what it means for a function $f\colon [a, b] \to \mathbb{R}$ to be absolutely continuous.
(2) Prove that if $f, g\colon [a, b] \to \mathbb{R}$ are absolutely continuous and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g$ is absolutely continuous.
(3) Prove that for any $\lambda, \mu \in \mathbb{R}$ there exists $f\colon [0, 1] \to \mathbb{R}$ such that
  (a) f is continuous,
  (b) $f(0) = 0$,
  (c) $f(1) = \lambda$,
  (d) for Lebesgue almost every x the derivative $f'(x)$ exists and is equal to μ.

  Hint: Consider a linear combination of the Cantor Lebesgue function and the identity function. Make sure to clearly state the properties of the Cantor Lebesgue function that you are using.
(4) Suppose that a function $f\colon [0, 1] \to \mathbb{R}$ is absolutely continuous and has properties (3)(b), (3)(c), (3)(d) listed above. Prove that $\lambda = \mu$.