Statistical Methods Prelim Exam

1:30 pm -4:00 pm, Wednesday, August 12, 2020

1. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the uniform distribution $U(\theta, \theta+1)$, $\theta \in \mathcal{R}$. Suppose $n \geq 2$.

   (a) Is there a unique minimum variance unbiased estimator (UMVUE) for θ? If yes, please derive its explicit form. If not, please carefully justify your anser.

   (b) Show the following test is a uniformly most powerful (UMP) size α test for $H_0 : \theta \leq 0$ versue $H_1 : \theta > 0$.

$$\phi(x) = \begin{cases} 0, & \min(X_1, X_2, \ldots, X_n) < 1 - \alpha^{\frac{1}{n}} \text{ or } \max(X_1, X_2, \ldots, X_n) < 1 \\ 1, & \text{otherwise.} \end{cases}$$

2. Suppose we have a single observation X from a Cauchy (θ) distribution, where the density of X is given by

$$p(x|\theta) = \frac{1}{\pi[1 + (x-\theta)^2]},$$

   for $-\infty < x < \infty$ and $-\infty < \theta < \infty$.

   (a) Suppose we wish to test $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$. Derive the likelihood ratio test (LRT) (simplest possible form) for this hypothesis, and find an explicit expression for the constant to make the test size α.

   (b) Compute the power function of the test in Part (a) and find the power at $\theta = 2$.

   (c) Is the test found in Part (a) unbiased? Justify your answer.

3. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from Poisson(θ) distribution, where θ is the mean parameter and $\theta > 0$ and $n \geq 1$.

   (a) Find the Jeffreys prior for θ and derive the Bayes estimator δ of θ under the squared error loss. Is δ an unbiased estimator?

   (b) Use the same Jeffreys prior derived in Part (a) and derive the Bayes estimator δ of θ under the scaled squared error loss $L_1(\theta, \delta) = E[(\theta - \delta)^2/\theta]$.

4. Assume a simple linear regression model $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$, $i = 1, \ldots, n$, where Y_i's are dependent variables and X_i's are independent variables, β_0 and β_1 are unknown parameters to be estimated, and the random errors ϵ_i are assumed following $N(0, \sigma^2)$ identically and indpendently with σ^2 unknown. Let $\theta = (\beta_0, \beta_1, \sigma^2)$.

   (a) Deirve the Fisher inforamtion matrix for θ.

   (b) Derive the UMVUE for θ. Please carefully justify the derived estimator is the UMVUE.