Preliminary Examination: LINEAR MODELS

Answer all questions and show all work.
Q1 is 30 points; Q2 is 35 points, and Q3 is 35 points.

1. Define

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix}, \text{ and } \boldsymbol{\beta} = \begin{pmatrix} \mu \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix}.$$

Assume the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $E(\boldsymbol{\varepsilon}) = \mathbf{0}$, and $var(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{I}$. Let $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ denote the columns of $\mathbf{X}$. *Note* that $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4 = \mathbf{0}$.

a. Find $E(\mathbf{Y})$ in terms of $\mu, \theta_1, \theta_2, \theta_3$, and θ_4.

b. Let $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)'$. Give the necessary and sufficient condition(s) for $\boldsymbol{\lambda}'\boldsymbol{\beta}$ to be estimable in the form of $\boldsymbol{\lambda}'\mathbf{c}_i = 0$, $i = 1, \ldots, s$. What is the value of s?

c. Is $\mu - \theta_1 + \theta_2 + \theta_3 - \theta_4$ estimable? Justify your answer.

2. Suppose that data $\{(x_{ij}, y_{ij}) : i = 1, \ldots, k, j = 1, \ldots, J\}$ can be modeled as having a common slope and possibly different intercepts using the linear model,

$$Y_{ij} = \beta_i + \gamma x_{ij} + \epsilon_{ij},$$

where $\{\epsilon_{ij}\}$ are independently and identically distributed $N(0, \sigma^2)$ random variables. Assume that no vector $(x_{i1}, \ldots, x_{iJ})$, for $i = 1, \ldots, k$, is proportional to the vector of 1s.

a. Determine the ordinary-least-squares estimator of $(\beta_1, \ldots, \beta_k, \gamma)'$.

b. Give an explicit expression for the size α likelihood-ratio test of the hypothesis,

$$H_0 : \beta_1 = \cdots = \beta_k = 0 \text{ versus } H_a : \text{ not } H_0$$

c. Compute the power of the test that you derived in part (b). Is the power dependent on γ?

3. Consider the following change-point model:

$$Y_i = \mu_i + e_i, \ i = 1, \dots, n,$$

where

$$\mu_i = \begin{cases} \beta_1, & \text{if } i \le n/2; \\ \beta_2, & \text{otherwise.} \end{cases}$$

Suppose we observe $Y_1, \dots, Y_n$. For simplicity, assume that the same size if even, namely $n = 2m$ for some integer $m > 0$.

a. Find the (ordinary) least squares estimator $(\hat{\beta}_1, \hat{\beta}_2)'$ for $(\beta_1, \beta_2)'$.

Assume that the errors satisfy:

$$e_i = \epsilon_i - a\epsilon_{i-1}, \ i = 1, \dots, n,$$

where $a \in \mathbb{R}$ is a parameter controlling the dependence strength, and ϵ_k, $k = 0, 1, \dots, n$, are independent normal random variables with mean zero and variance $\sigma^2 > 0$.

b. For parts (b) and (c) only, assume that $a = 0$. Find the joint distribution of $(\hat{\beta}_1, \hat{\beta}_2)'$. Are $\hat{\beta}_1$ and $\hat{\beta}_2$ independent in this case? How does the variance of $\hat{\beta}_1$ change when $n \to \infty$ (for example whether it decreases to zero linearly in n, quadratically or at some other rate)?

c. Following part (b), find an unbiased estimator $\hat{\sigma}^2$ of σ^2 and devise a statistically valid test for the following hypotheses:

$$H_0 : \beta_1 = \beta_2 \text{ vs } H_a : \text{ Not } H_0$$

You need to specify the test statistic and its distribution under the null hypothesis.

d. Now suppose that $a = 1$, find the joint distribution of $(\hat{\beta}_1, \hat{\beta}_2)'$. Are $\hat{\beta}_1$ and $\hat{\beta}_2$ independent in this case? How does the variance of $\hat{\beta}_1$ change when $n \to \infty$ in this case (for example whether it decreases to zero linearly in n, quadratically or at some other rate)? Compare your result with the one in part (b) and comment on the effect of dependence among the errors. Is dependence always a "bad" thing?

e. Now suppose that $0 < a < 1$. Find the distribution of $\hat{\beta}_1$. How does the variance of $\hat{\beta}_1$ change when $n \to \infty$ (for example whether it decreases to zero linearly in n, quadratically or at some other rate)? Compare your result with the ones in parts (b) and (d) and comment.