PRELIMINARY EXAM

PARTIAL DIFFERENTIAL EQUATIONS AUGUST 16, 2021

FULL NAME:

ID NUMBER:

Instruction: Choose **only five** (out of six) problems to do. Each problem is worth 20 points.

Scores:

Problem 1	____________	Problem 2	____________
Problem 3	____________	Problem 4	____________
Problem 5	____________	Problem 6	____________

Total ____________

1. Let $D \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Show that any smooth solution u of the nonlinear problem

$$\begin{cases} \Delta u + u^2(1-u) = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

satisfies $0 \leq u(x) \leq 1$ for all $x \in \overline{D}$.

2. Let D be the open disk $D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 4\}$ and suppose that $u \in C^2(D) \cap C(\overline{D})$ is harmonic in D, satisfying $u(0, 0) = 2$ and $u(0, 1) = 10$. Show that $u(x, y)$ cannot be non-negative for all (x, y) satisfying $x^2 + y^2 \leq 4$.

3. Let $L > 0$ and $T > 0$ be fixed. Assume that $u(x,t)$ is a smooth function that satisfies

$$\begin{cases} u_t - u_{xx} + c(x,t)u = 0 & \text{for } (x,t) \in (0,L) \times (0,T), \\ u(0,t) \geq 0 \text{ and } u(L,t) \geq 0 & \text{for } 0 < t < T, \\ u(x,0) \geq 0 & \text{for } 0 < x < L, \end{cases}$$

where $c(x,t)$ is any function satisfying $|c(x,t)| \leq M$ for all $(x,t) \in (0,L) \times (0,T)$, and some constant $M > 0$. Show that

$$u(x,t) \geq 0 \quad \text{for } (x,t) \in (0,L) \times (0,T).$$

4. Let $T > 0$ be a given terminal time, $L > 0$, and h be a continuous function on $[0, L]$. Consider the following initial-boundary-value problem

$$\begin{cases} u_{tt} - u_{xx} + u^3 = 0 & \text{for } (x,t) \in (0,L) \times (0,T], \\ u(0,t) = 0, \quad u(L,t) = 0 & \text{for } t \in [0,T], \\ u(x,0) = 0, \quad u_t(x,0) = h(x) & \text{for } x \in (0,L). \end{cases} \tag{$*$}$$

Suppose that $u = u(x,t)$ is a smooth solution of the problem $(*)$. Show that for each $t > 0$,

$$\int_U \frac{1}{2}\left[u_t(x,t)\right]^2 + \frac{1}{2}|u_x(x,t)|^2 \, \mathrm{d}x \le \int_U \frac{1}{2}[h(x)]^2 \, \mathrm{d}x.$$

Hint: Multiply the equation by u_t and use an energy argument.

5. Let $\varepsilon > 0$ and define for $x \in \mathbb{R}$, $g(x) = \begin{cases} 1 & x < -\varepsilon, \\ \dfrac{\varepsilon - x}{2\varepsilon} & -\varepsilon \leq x \leq \varepsilon, \\ 0 & x > \varepsilon. \end{cases}$

(a) Construct the entropy solution of the initial-value problem

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

(b) Study what happens to the solution you constructed as $\varepsilon \to 0^+$.

6. Let $\Omega = \{(x,y) \in \mathbb{R}^2 \colon y < x^2 + x,\ x > 0,\ y > 0\}$. Use the method of characteristics to find the solution $u(x,y)$ of the following problem

$$\begin{cases} u_x + u_y + u = 1 & \text{for } (x,y) \in \Omega, \\ u(x, x + x^2) = \sin(x) & \text{for } x > 0. \end{cases}$$