1. Suppose $X_1, X_2, \ldots$ are independent random variables such that
$$\mathbb{P}(X_n = n) = \frac{1}{2n}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n}, \quad \mathbb{P}(X_n = -1) = \frac{1}{2n}, n \in \mathbb{N}.$$
   (a) State the Kolmogorov's three-series theorem.
   (b) Apply the three-series theorem to determine for what values of θ the series $\sum_{n=1}^{\infty} \frac{X_n}{n^\theta}$ converges almost surely.

2. Let X_1, X_2, ... be a sequence of i.i.d. random variables with
$$\mathbb{P}(X_1 > x) = \frac{2x}{x^4 + 1}, x \in \mathbb{N}.$$
Show that for some $\beta > 0$, we have $n^{-\beta} \max_{i=1,\ldots,n} X_i$ converges weakly to a non-degenerate distribution. Identify β and the limit distribution.

3. Let X_1, X_2, ... be a sequence of independent random variables such that
$$\mathbb{P}(X_k = k^\lambda) = \frac{1}{2} \quad \text{and} \quad \mathbb{P}(X_k = -k^\lambda) = \frac{1}{2}, k \in \mathbb{N}.$$
Write $S_n := \sum_{i=1}^n X_i, n \in \mathbb{N}$. Throughout, assume $\lambda > 1/2$.
   (a) Show that for all $\lambda > -1/2$
$$\lim_{n\to\infty} \frac{\mathbb{E}(S_n^2)}{n^{2\lambda+1}} = \frac{1}{2\lambda + 1}.$$
   (b) Find a sequence of $\{a_n\}_{n\in\mathbb{N}}$ such that
$$\frac{S_n}{a_n} \Rightarrow N(0, 1).$$

4. Let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables such that, for some fixed $p \in (0, 1)$,
$$\mathbb{P}\left(Y_1 = \frac{2}{3}\right) = 1 - \mathbb{P}\left(Y_1 = \frac{5}{4}\right) = p.$$
Set $X_n := \prod_{i=1}^n Y_i, n \in \mathbb{N}$.
   (a) Does the limit of $\mathbb{E}X_n$ exist as $n \to \infty$?
   (b) Does the limit of $X_n^{1/n}$ exist as $n \to \infty$?

   Justify your answers, which may depend on the values of p. For the second part, specify the mode of convergence regarding your answer.