REAL ANALYSIS PRELIMINARY EXAM
AUGUST 2021

Time allowed: 2 hours 30 minutes.
Answer all problems and fully justify your work.
m denotes Lebesgue measure and $\mathcal{M}$ Lebesgue measurable sets.
Unless otherwise specified, measurable means with respect to $\mathcal{M}$.

Question 1

(a) Show that if $f, g\colon \mathbb{R} \to \mathbb{R}$ are measurable functions, then $f + g$ is also measurable.
(b) Give an example of function $f\colon \mathbb{R} \to \mathbb{R}$ which is measurable but nowhere continuous. Briefly support your claims.
(c) Suppose $E \subset \mathbb{R}$ is measurable with $m(E) > 0$. Must E contain an interval of positive length? Give a proof or counterexample.
(d) Suppose $E \subset \mathbb{R}$ is measurable and contains an interval of positive length. Must $m(E) > 0$? Give a proof or counterexample.

Question 2

(a) Suppose $f_n\colon [0,1] \to \mathbb{R}$ is a sequence of integrable functions converging pointwise to an integrable function $f\colon [0,1] \to \mathbb{R}$. Is it necessarily true that $\int_0^1 f_n \,\mathrm{d}m \to \int_0^1 f \,\mathrm{d}m$? Give a proof or counterexample.
(b) Carefully justifying your argument, evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{n^2 \cos x}{1 + n^3\sqrt{x}} \,\mathrm{d}x.$$

Question 3

(a) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable functions. Show that
$$\{(x,t) \in \mathbb{R}^2 : \min(f(x), g(x)) \leq t < \max(f(x), g(x))\}$$
is a Lebesgue measurable subset of $\mathbb{R}^2$.

(b) Let $F_t = \{x \in \mathbb{R} : f(x) > t\}$ and $G_t = \{x \in X : g(t) > t\}$. Show that
$$(F_t \setminus G_t) \cup (G_t \setminus F_t) = \{x \in \mathbb{R} : \min(f(x), g(x)) \leq t < \max(f(x), g(x))\}.$$

(c) Show that $\int |f - g| \, \mathrm{d}m = \int_{-\infty}^{\infty} m((F_t \setminus G_t) \cup (G_t \setminus F_t)) \, \mathrm{d}t$.

Question 4

(a) Give the definition of what it means for a function $f\colon [0,1] \to \mathbb{R}$ to be absolutely continuous.

(b) Define $g\colon [0,1] \to \mathbb{R}$ by $g(0) = 0$ and $g(x) = x \cos \frac{\pi}{x}$ if $x \neq 0$. Show that g is continuous on $[0,1]$ but not absolutely continuous.

(c) Let $f\colon [0,1] \to \mathbb{R}$ be absolutely continuous and $E \subset [0,1]$ such that $m(E) = 0$. Show that $m(f(E)) = 0$.