Statistical Methods Prelim Exam

1:00 pm - 3:30 pm, Friday, August 20, 2021

1. Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X which has a $N(\mu, \sigma^2)$ distribution, where σ^2 is known. Let $\bar{X}$ represent the sample mean. Consider the two-sided hypotheses

$$H_0 : \mu = 0 \text{ versus } H_1 : \mu \neq 0.$$

   (a) Consider the test with the critical function

$$\phi_1(x) = \begin{cases} 1, & \bar{X} > z_\alpha \cdot \sigma/\sqrt{n}; \\ 0, & \text{otherwise,} \end{cases}$$

   where z_α represents the upper 100α-percentile of a standard normal distribution. Show that the test based on $\phi_1(x)$ has a size of α but it is ***not*** an unbiased test.

   (b) Consider the test with the following critical function

$$\phi_2(x) = \begin{cases} 1, & |\bar{X}| > z_{\alpha/2} \cdot \sigma/\sqrt{n}; \\ 0, & \text{otherwise.} \end{cases}$$

   Show that the test based on $\phi_2(x)$ is an unbiased size α test. (Hint: Check the monotonicity of the power function.)

2. Let $X_1, \ldots, X_n$ be iid random sample from $U(\theta, \theta + 1)$, where $-\infty < \theta < \infty$ and it is unknown. Assume a prior distribution for θ given by the probability density function, for $-\infty < \theta < \infty$,

$$\pi(\theta) = \frac{1}{2} e^{-|\theta|} .$$

   Find the Bayes estimate of θ with respect to the squared error loss (SEL) function.

3. Let $X_1, ..., X_n$ be a random sample from $N(\mu_1, \sigma^2)$, and let $Y_1, ..., Y_n$ be a random sample from $N(\mu_2, \sigma^2)$, independent of the previous sample. Assume μ_1 and μ_2 are unknown, $-\infty < \mu_1, \mu_2 < \infty$, and $\sigma^2 > 0$ is known.

   (a) Find the UMVUE of $\eta = P(X < Y)$ where X and Y are independent with distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, respectively, which are the same as the ones given above.

   (b) Is the UMVUE of η a consistent estimator?

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population having the following density

$$f(x; \theta_1, \theta_2) = \begin{cases} (\theta_1 + \theta_2)^{-1} \exp\left(-x/\theta_1\right), & x > 0 \\ (\theta_1 + \theta_2)^{-1} \exp\left(x/\theta_2\right), & x \leq 0, \end{cases}$$

   where $\theta_1 > 0$ and $\theta_2 > 0$ are unknown.

   (a) Show in detail that the maximum likelihood estimator (MLE) of $\theta = (\theta_1, \theta_2)$ is

$$\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2) = n^{-1}(\sqrt{T_1 T_2} + T_1, \sqrt{T_1 T_2} + T_2),$$

   where $T_1 = \sum_{i=1}^{n} X_i I(0 < X_i < \infty)$ and $T_2 = -\sum_{i=1}^{n} X_i I(-\infty < X_i < 0)$.

   (b) Obtain a nondegenerated asymptotic distribution of the MLE of θ.

   (c) Find a likelihood ratio (LR) test of size α for testing $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$. Clearly write down the likelihood functions under H_0, H_1, the likelihood ratio, and the decision rule.