Statistical Methods Prelim Exam

1:00 pm - 3:30 pm, Thursday, January 7, 2021

1. Let $X_1, \ldots, X_n$ be a sample from the *inverse Gaussian* distribution, denoted by $I(\mu, \tau)$, with its density
$$\sqrt{\frac{\tau}{2\pi x^3}} \exp\left[-\frac{\tau}{2x\mu^2}(x-\mu)^2\right], \quad x > 0, \quad \tau, \mu > 0.$$
   (a) Find the moment generating function of X_1 and show that $V_1 = \frac{\tau}{\mu^2 X_1}(X_1 - \mu)^2 \sim \chi_1^2$.
   (b) Show that $\overline{X} = \sum_{i=1}^n X_i/n \sim I(\mu, n\tau)$.
   (c) Show that there exists a uniformly most powerful (UMP) test for testing $H_0 : \mu \leq \mu_0$ versus. $H_1 : \mu > \mu_0$ when τ is known.
   (d) Show that there exists a UMP test for testing $H_0 : \tau \leq \tau_0$ versus. $H_1 : \tau > \tau_0$ when μ is known.

2. The random variable X has a probability distribution given by

x	0	1	2	3	4	$\cdots$
$p(x)$	p	q^2	q^2p	q^2p^2	q^2p^3	$\cdots$

   where $0 < p < 1, q = 1 - p$. (Note: a result that you may need is $\sum_{i=0}^{\infty} ip^i = p/(1-p)^2$.)
   (a) Let U be an unbiased estimator of zero. Show that $U(X+1) = -XU(0), X = 0, 1, 2, 3, \ldots$.
   (b) Show that there is **no** uniformly minimum variance unbiased estimator (UMVUE) for p.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with the probability density function specified below,
$$p_\theta(x) = a^\theta \theta x^{-\theta-1}, \quad x \geq a,$$
   where $\theta > 0$ is unknown and $a > 0$ is a known constant.
   (a) Find the maximum likelihood estimator $(\hat{\theta})$ of θ.
   (b) Find the asymptotic distribution $\sqrt{n}(\hat{\theta} - \theta)$.
   (c) Show $\hat{\theta}$ is a biased estimator for θ. Derive the UMVUE $\tilde{\theta}$ of θ.
   (d) Find the asymptotic distribution of $\sqrt{n}(\tilde{\theta} - \theta)$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample of binary random variables with $P(X_i = 1) = p$, $i = 1, \ldots, n$ and $0 < p < 1$.
   (a) Obtain the Bayes estimator of $p(1-p)$ when the prior is the beta distribution with known parameter (α, β), under the squared error loss.
   (b) Discuss the bias and consistency of the Bayes estimator obtained in (a).
   (c) Let $[p(1-p)]^{-1}I(0 < p < 1)$ be an improper prior density for p. Show that the posterior of p given X_i's is a (proper) probability density provided that the sample mean $\bar{X} \in (0, 1)$.
   (d) Under the squared error loss, find the Bayes estimator of $p(1-p)$ under the improper prior given in (c).