# Complex Analysis Prelim Exam
## UC Department of Math
## Jan 2021

(1) Use methods of complex variables to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx.$$

(2) (a) Assume the infinite series $\sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$ and let $f(z)$ be the limit. Show that for $r < R$,

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

(b) Deduce Liouville's theorem from (a).

*Recall Liouville's theorem says: If $f(z)$ is entire and bounded, then $f$ is constant.*

(3) Find all $v : \mathbb{R}^2 \to \mathbb{R}$ such that for $z = x + iy$, $f(z) = (x^3 - 3xy^2) + iv(x, y)$ is analytic.

(4) Suppose $f$ has an isolated singularity at $a \in \mathbb{C}$ and $|\mathrm{Re} f| \leq M$ on $D(a, R) \setminus \{a\}$ for some $R > 0$ and $M \geq 0$. Prove that the singularity is removable.

(5) Let $P(z)$ and $Q(z)$ be complex polynomials of degrees $n$ and $m$ respectively. Define

$$F(z) := \frac{P(z)}{Q(z)}.$$

(a) Exhibit a finite set $S \subset \mathbb{C}$ such that $F$ is holomorphic at each point $z$ in $\mathbb{C} \setminus S$, and explain why each point in $S$ is an isolated singularity of $F$.
(b) Describe the nature of the singularity of $F = P/Q$ at each point $a$ in $S$, and explain how you would compute the residue of $F = P/Q$ at $a$.
(c) Assuming $R$ is sufficiently large, describe how to calculate each of the following:

$$\int_{|z|=R} F(z)\,dz \qquad \text{and} \qquad \int_{|z|=R} \frac{F'(z)}{F(z)}\,dz\,.$$