Ordinary Differential Equations Preliminary Exam May 2021

1. Prove the following property of the matrix exponential:

$$\frac{d}{dt} \left[e^{\mathbf{A}t} \right] = \mathbf{A}e^{\mathbf{A}t},$$

where $\mathbf{A} \in \mathbb{C}^{N \times N}$ and $t \in \mathbb{C}$.

2. Consider the equation

$$\dot{x} = rxe^{-x^2} - x\left(1 + x^2\right).$$

- (a) Sketch the phase line, identify the equilibrium, and determine stability for representative values of r.
- (b) Draw the bifurcation diagram and identify the type of bifurcation as r varies.
- 3. Consider the system

$$\dot{x} = x - y - (x + y) (x^2 + y^2)$$

 $\dot{y} = x + y + (x - y) (x^2 + y^2)$

- (a) Determine all the equilibria and their stability.
- (b) Show there exists a stable periodic orbit and state where.
- 4. Consider the equation

$$\ddot{x} = \left(1 - x^2\right)\left(4 - x^2\right).$$

- (a) State the energy function $E(x, \dot{x})$.
- (b) State and classify the equilibria.
- (c) Sketch the phase portrait in detail.
- 5. Consider the system

$$\dot{x} = ax + by
\dot{y} = cx + dy$$

with constant a, b, c, and d. Assume that a+d<0 and ad-bc>0. Show that all solutions tend to zero as $t\to\infty$.