Statistical Methods Prelim Exam

1:00 pm - 3:30 pm, Wednesday, May 5, 2021

1. Let us denote $\mathbf{X}^T = (X_1, X_2)$ where $\mathbf{X}$ is assumed to have the bivariate normal distribution, $N_2(\mu, \mu, 1, 1, \sqrt{2})$. Hence, we consider $\mu(\in \mathbf{R})$ as the unknown parameter. With preassigned $\alpha \in (0, 1)$, derive a level α Likelihood ratio (LR) test for $H_0 : \mu = \mu_0$ v.s. $H_1 : \mu \neq \mu_0$ where μ_0 is a real number, in the implementable form. Note that the joint pdf of (X_1, X_2) is given by

$$f(x_1, x_2; \mu) = \frac{1}{\pi\sqrt{2}} \exp[-(x_1 - \mu)^2 + \sqrt{2}(x_1 - \mu)(x_2 - \mu) - (x_2 - \mu)^2] \, .$$

(Make sure there is only one data point available here.)

2. Let X be a discrete random variable with

$$P_\theta(X = x) = 1/\theta \text{ for } x = 1, ..., \theta.$$

where θ is an unknown positive integer.

(a) Let $\Theta = \{$ all positive integers $\}$ be the parameter space for θ. Show that the family of distributions X is complete and find the UMVUE for θ.

(b) Assume now that the parameter space is $\Theta_1 = \{\theta \in \Theta : \theta \neq k\}$, where k is a known positive integer. Show that the family of distributions of X is no longer complete, and show that the UMVUE in part (a) is no longer the UMVUE.

3. Suppose $X_1, \ldots, X_n$ are iid with density

$$p(x|\theta_1, \theta_2) = \frac{1}{2\theta_2} \exp(-|x - \theta_1|/\theta_2),$$

where θ_1 is known, θ_2 is unknown, $-\infty < \theta_1 < \infty$, $\theta_2 > 0$, and $-\infty < x < \infty$.

(a) Show that $p(x|\theta_1, \theta_2)$ has the monotone likelihood ratio(MLR) property in some statistic $T(\mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_n)$, and identify $T(\mathbf{x})$.

(b) Derive the distribution of $Y_i = |X_i - \theta_1|$.

(c) Consider testing $H_0 : \theta_2 \leq \theta_0$ against $H_1 : \theta_2 > \theta_0$. Find a UMP size α test. Express your test in the simplest possible form and find the simplest expression for the constant to make the test size α.

4. Let $\bar{X}$ be the sample mean of a random sample of size n from $N(\theta, \sigma^2)$ with a known $\sigma > 0$ and an unknown $\theta \in \mathcal{R}$. Let $\pi(\theta)$ be a prior density with respect to a σ-finite meaure ν on $\mathcal{R}$.

(a) Show that the posterior mean of θ, given $\bar{X} = x$, is of the form

$$\delta(x) = x + \frac{\sigma^2}{n} \frac{d \log(p(x))}{dx},$$

where $p(x)$ is the marginal density of $\bar{X}$, unconditional on θ.

(b) Show that the posterior variance of θ (given $\bar{X} = x$) is

$$\text{Var}(\theta|\bar{X} = x) = \frac{\sigma^4}{n^2} \frac{d^2 \log p(x)}{dx^2} + \frac{\sigma^2}{n}.$$