PRELIMINARY EXAM IN ALGEBRA - AUGUST 2022

Time allowed : 2 hours and 30 minutes
No calculators, cell phones, or other electronic devices allowed

1. Prove or disprove:
   (a) $\mathbb{Z}[x]$ is a principal ideal domain.
   (b) Let $R = \mathbb{Z}[\sqrt{2}]$ and Q be the field of fractions of R. Then $Q \cong \mathbb{Q}[\sqrt{2}]$.

2. Prove that the following functions are irreducible in $\mathbb{Z}[x]$:
   (a) $f(x) = x^4 + 9x^3 + 6x^2 + 15x + 3$
   (b) $g(x) = x^3 - 9x + 9$

3. Let E/F be an algebraic field extension.
   (a) What does it mean for E to be a splitting field for some $f(x) \in F[x]$?
   (b) What does it mean for E/F to be a normal extension?
   (c) Prove that if E is a splitting field then it is normal.

4. Let F be a field of characteristic zero, and let $\alpha_1, \dots, \alpha_4$ be indeterminates — or you can assume they are algebraically independent elements over F. Let
   $$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$
   and let $L = F(\alpha_1, \alpha_2, \alpha_3, \alpha_4), K = F(a_0, a_1, a_2, a_3)$.
   (a) Show that L/K is Galois. What is the Galois group?
   (b) Use the structure of L/K to demonstrate a degree 4 extension M/N such that there exists no quadratic extension of N contained in M.

5. Let $R = \mathbb{Z}[x]$. Prove that any strictly ascending sequence of ideals $I_1 \subsetneq I_2 \subsetneq \dots$ of R must be finite.