# Preliminary Examination: LINEAR MODELS

Answer all questions and show all work.
Q1 is 30 points; Q2 is 35 points, and Q3 is 35 points.

1. Let $\mathbf{Y}$ be an $n$-dimensional response vector. $\mathbf{X}_1$ and $\mathbf{X}_2$ are $n \times p$ and $n \times q$ matrices, respectively. Suppose that the correct model is:

$$\mathbf{Y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon},$$

where $E(\boldsymbol{\varepsilon}) = \mathbf{0}$ and $\text{var}(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{I}$. Suppose that we fit the following *incorrect* model:

$$\mathbf{Y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}$$

with the same assumptions for $\boldsymbol{\varepsilon}$. Assume that $\mathbf{X}_1$ is full rank.

a. Consider the ordinary least squares (OLS) estimator $\hat{\boldsymbol{\beta}}_1$ of $\boldsymbol{\beta}_1$ under the correct model. Let $\tilde{\boldsymbol{\beta}}_1$ denote the OLS estimator of $\boldsymbol{\beta}_1$ when we fit the *incorrect* model. Assuming in part (a) that $\boldsymbol{\beta}_2 = \mathbf{0}$, compare $\hat{\boldsymbol{\beta}}_1$ and $\tilde{\boldsymbol{\beta}}_1$ in terms of bias and variance.

Hint:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{C} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0}' & \mathbf{0} \end{pmatrix} + \begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} \\ \mathbf{I} \end{pmatrix} (\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}(-\mathbf{B}'\mathbf{A}^{-1} \;\; \mathbf{I})$$

b. Assume in part (b) that $\boldsymbol{\varepsilon} \sim \mathcal{N}_n(\mathbf{0}, \sigma^2\mathbf{I})$. When we fit the incorrect model and would like to test $H_0 : \boldsymbol{\beta}_1 = \mathbf{0}$ vs. $H_a : \boldsymbol{\beta}_1 \neq \mathbf{0}$, suppose we use the usual F test statistics assuming the *incorrect* model:

$$F = \frac{SSM/p}{SSE/(n-p)}.$$

Give the expressions of the model sum of squares ($SSM$) and residual sum of squares ($SSE$). What are the actual distributions of $SSM$ and $SSE$ under $H_0$ (and the correct model), respectively? Comment on the validity of this F test for $H_0$: $\boldsymbol{\beta}_1 = \mathbf{0}$.

2. Consider the normal linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where $\mathbf{X}$ is an $n \times p$ design matrix with full rank and $\boldsymbol{\varepsilon} \sim \mathcal{N}_p(\mathbf{0}, \sigma^2 \mathbf{I}_n)$. Let $\hat{\boldsymbol{\beta}}$ be the ordinary least squares (OLS) estimator of $\boldsymbol{\beta}$. The ridge regression estimator $\hat{\boldsymbol{\beta}}_R(\lambda)$ of $\boldsymbol{\beta}$ is the vector-value of $\boldsymbol{\beta}$ that minimizes

$$Q(\boldsymbol{\beta}) = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) + \lambda \boldsymbol{\beta}'\boldsymbol{\beta},$$

where $\lambda > 0$ is a fixed real number.

a. Find an expression for $\hat{\boldsymbol{\beta}}_R(\lambda)$. Show that you can write $\hat{\boldsymbol{\beta}}_R(\lambda) = \mathbf{W}_\lambda \hat{\boldsymbol{\beta}}$ for some matrix $\mathbf{W}_\lambda$ that depends on $\lambda$ and give the explicit expression for $\mathbf{W}_\lambda$.

b. Find the mean, bias and variance of $\hat{\boldsymbol{\beta}}_R(\lambda)$.

c. Show that $Var(\hat{\boldsymbol{\beta}}_R(\lambda)) < Var(\hat{\boldsymbol{\beta}}))$ in the sense that $Var(\hat{\boldsymbol{\beta}}) - Var(\hat{\boldsymbol{\beta}}_R(\lambda))$ is a positive definite matrix. Hint: You may use $\mathbf{X}'\mathbf{X} = \mathbf{PDP}'$ where $\mathbf{D}$ is a diagonal matrix and $\mathbf{P}$ is an orthogonal matrix.

d. Find the mean squared error of $\hat{\boldsymbol{\beta}}_R(\lambda)$, $MSE(\lambda) = E\{(\hat{\boldsymbol{\beta}}_R(\lambda) - \boldsymbol{\beta})'(\hat{\boldsymbol{\beta}}_R(\lambda) - \boldsymbol{\beta})\}$.

e. Assuming now that $\mathbf{X}'\mathbf{X} = d\,\mathbf{I}_p$ where $d$ is a fixed constant, find the optimum value of $\lambda$ that minimizes the $MSE(\lambda)$.

3. This problem concerns the situation where doubts are casted on the stability assumption of the regression coefficient and the independence assumption of the errors. Consider the following change-point model:

$$Y_i = \mu_i + \epsilon_i, \;\; i = 1, \ldots, n,$$

where

$$\mu_i = \begin{cases} \beta_1, & \text{if } i \le n/2; \\ \beta_2, & \text{otherwise.} \end{cases}$$

Suppose that we only observe $Y_1, \ldots, Y_n$. For simplicity, assume that the sample size is even, namely $n = 2m$ for some integer $m > 0$. Let $(\hat{\beta}_1, \hat{\beta}_2)$ be the ordinary least squares (OLS) estimator of $(\beta_1, \beta_2)$.

a. Find the OLS estimator $(\hat{\beta}_1, \hat{\beta}_2)$.

Assume that the errors satisfy

$$\epsilon_i = e_i - a\, e_{i-1}, \;\; i = 1, \ldots, n,$$

where $a \in \mathcal{R}$ is a parameter controlling the dependence strength, and $e_k, k = 0, 1, \ldots, n$, are independent normal random variables with mean zero and variance $\sigma^2 > 0$.

b. Assume that $a = 0$. Find the joint distribution of $(\hat{\beta}_1, \hat{\beta}_2)$. Are $\hat{\beta}_1$ and $\hat{\beta}_2$ independent in this case? Howe does the variance of $\hat{\beta}_1$ change when $n \to \infty$ (for example, whether it decreases to zero linearly in $n$, quadratically or at some other rate)?

c. Assume that $a = 0$. Find an unbiased estimator of $\hat{\sigma}^2$ of $\sigma^2$ and derive a test for:

$$H_0: \;\beta_1 - \beta_2 = 0 \text{ vs } H_a: \beta_1 - \beta_2 \neq 0$$

You need to specify the test statistic and its distribution under the null hypothesis.

d. Now assume that $a = 1$. Find the joint distribution of $(\hat{\beta}_1, \hat{\beta}_2)$. Are $\hat{\beta}_1$ and $\hat{\beta}_2$ independent in this case? How does the variance of $\hat{\beta}_1$ change when $n \to \infty$ in this case (for example whether it decreases to zero linearly in $n$, quadratically or at some other rate)? Compare your result with the one in part (b) and comment on the effect of dependence among the errors. Is dependence always a "bad" thing?

e. Now suppose that $0 < a < 1$. Find the distribution of $\hat{\beta}_1$. How does the variance of $\hat{\beta}_1$ change when $n \to \infty$ (for example whether it decreases to zero linearly in $n$, quadratically or at some other rate)? Compare your result with the ones in parts (b) and (d) and comment.