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MATHEMATICS QUALIFYING EXAM, AUGUST 16, 2022

Four Hour Time Limit

In this exam $\mathbb{R}$ denotes the field of all real numbers, $\mathbb{R}^n$ is n-dimensional Euclidean space with the usual dot product $\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^{m} u_i v_i$. Proofs, or counter examples, are required for all problems.

1. Let (X, d) be a metric space and $\gamma : [0,1] \to X$ be a continuous function such that $\gamma(0) \neq \gamma(1)$. Let $f : X \to \mathbb{R}$ be a continuous function such that $f(\gamma(0)) = 1$ and $f(\gamma(1)) = 0$. Prove that $[0,1] \subset f(X)$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ for rational x and $f(x) = -x$ for irrational x. Prove that f is not Riemann integrable on the interval $[0,1]$.

3. Let $a < b$ and $G : (a,b) \to \mathbb{R}$, and let $a < x_0 < b$. Prove that if there are two real numbers $\alpha > 0$ and $\tau > 1$ such that for each $x \in (a,b)$ we have $|G(x) - G(x_0)| \leq \alpha\, |x - x_0|^\tau$, then G is differentiable at x_0.

4. Set $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{\cos(kx)}{k^2+1}$ for $x \in \mathbb{R}$. Prove that f is well defined and continuous on $\mathbb{R}$.

5. Let V be a vector space, and $U, W \subset V$ be subspaces such that $U \cap W = \{0\}$ and $U + W = V$. Let T be a linear transformation such that $Tu = u$ for all $u \in U$, and $Tw = 2w$ for all $w \in W$. Prove that $T^2 v + 2v = 3Tv$ for all $v \in V$.

6. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation that preserves lengths, i.e., $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.
(a) Show that T preserves orthogonality, i.e., if $\mathbf{x} \cdot \mathbf{y} = 0$, then $T(\mathbf{x}) \cdot T(\mathbf{y}) = 0$.
(b) Show that the columns of the matrix representation of T in the standard basis of $\mathbb{R}^n$ are mutually orthogonal.

7. Let $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$. Derive the formula for the enries of the matrix A^n as explicit functions of $n \geq 3$.
(entries)

8. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $c \in \mathbb{R}^n$.
(a) Give the definition of the derivative $Df(c)\colon \mathbb{R}^n \to \mathbb{R}^m$.
(b) Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Use the definition to calculate the derivative $DL(c)$ for every $c \in \mathbb{R}^n$.