PRELIMINARY EXAM

PARTIAL DIFFERENTIAL EQUATIONS JANUARY 5, 2022

FULL NAME:

ID NUMBER:

Instruction: Choose **only five** (out of six) problems to do. Each problem is worth 20 points.

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	20	20	100
Score:							

1. [20 points] Let $U \subset \mathbb{R}^n$, $n \geq 2$, be open, bounded, and connected with C^1 boundary. Consider the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } U, \\ \dfrac{\partial u}{\partial \nu} + \alpha u = \beta & \text{on } \partial U, \end{cases}$$

where α and β are constants, with $\alpha > 0$; and ν denotes the outward unit normal vector field on the boundary ∂U. Prove that this problem has **at most one** smooth solution $u = u(x)$.

2. [20 points] Let Ω be the open disk $\Omega = \{(x, y)\colon x^2 + y^2 < 9\}$ and suppose that g is a continuous non-negative function, i.e. $g(x, y) \geq 0$, defined on the circle $\partial\Omega = \{(x, y)\colon x^2 + y^2 = 9\}$. Prove that there is **no** smooth solution of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

with the values $u(0, 0) = 1$ and $u(0, 1) = 3$.

3. [20 points] Find the solution $u(x, t)$ of the following initial-value problem

$$\begin{cases} u_{tt} - 9u_{xx} = x, & -\infty < x < \infty, \quad t > 0, \\ u(x, 0) = 0, & -\infty < x < \infty, \\ u_t(x, 0) = 0, & -\infty < x < \infty. \end{cases}$$

4. [20 points] Let $L > 0$ and $T > 0$ be given. Assume that $u(x, t)$ satisfies

$$\begin{cases} u_t - u_{xx} + c(x, t)u \geq 0, & 0 < x < L, \quad 0 < t < T, \\ u(x, 0) \geq 0, & 0 < x < L, \\ u(0, t) \geq 0, & 0 < t < T, \\ u(L, t) \geq 0, & 0 < t < T, \end{cases}$$

where $c(x, t)$ is *any* function satisfying $|c(x, t)| \leq M$ for all $0 < x < L$ and $0 < t < T$, and some constant $M > 0$. Show that

$$u(x, t) \geq 0 \qquad \text{for } 0 < x < L \text{ and } 0 < t < T.$$

5. [20 points] Use the method of characteristics to find the solution $u(x, t)$ of the following problem

$$\begin{cases} xu_x + (t + 1)u_t = u, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases}$$

6. [20 points] Find the entropy solutions of the following problems

$$\begin{cases} u_t + u^5 u_x = 0, & t > 0, \\ u(x, 0) = g(x), \end{cases}$$

with

(a) $g(x) = \begin{cases} 3, & x < 0, \\ 1, & x > 0. \end{cases}$

(b) $g(x) = \begin{cases} 1, & x < 0, \\ 3, & x > 0. \end{cases}$