1. Prove the following property of the matrix exponential:

$$
e^{\mathbf{A}} \leq e^{\|\mathbf{A}\|}
$$

2. Consider the equation

$$
\dot{x}=r^{2} x^{2}-2 x+r
$$

for $r \in \mathbb{R}$ with $r>0$.
(a) Sketch the phase line, identify the equilibrium, and determine stability for representative values of $r$.
(b) Draw the bifurcation diagram and identify the type of bifurcation as $r$ varies.
3. A gradient system for $\boldsymbol{x}=\left[x_{1}, x_{2}, \ldots, x_{n}\right]^{T} \in \mathbb{R}^{n}$ is a system of the form

$$
\boldsymbol{x}^{\prime}=-\nabla V(\boldsymbol{x}) \quad \text { or } \quad x_{i}^{\prime}=-\frac{\partial V}{\partial x_{i}} \text { for } i=1,2, \ldots, n
$$

for some smooth function $V: \mathbb{R}^{n} \rightarrow \mathbb{R}$.
Suppose $\boldsymbol{x}(t)$ is a solution of the gradient system with initial condition $\boldsymbol{x}(0)=\boldsymbol{x}_{\mathbf{0}}$.
(a) Prove that $V(\boldsymbol{x}(t)) \leq V\left(\boldsymbol{x}_{\mathbf{0}}\right)$
(b) Show that

$$
x^{\prime}=-x+2 y-x^{3} \quad \text { and } \quad y^{\prime}=2 x-y-y^{3}
$$

is a gradient system by finding an appropriate function $V(x, y)$.
4. Assume that the function $u(x) \geq 0$ is of class $C[1, \infty)$, and

$$
x u(x) \leq K+\int_{1}^{x} u(t) d t \text { for } x \geq 1
$$

Show that $u(x) \leq K$ for $x \geq 1$.
5. Consider the system

$$
\begin{aligned}
x^{\prime} & =y-y^{3} \\
y^{\prime} & =-\left(x+y^{2}\right)
\end{aligned}
$$

(a) Determine all equilibria and classify.
(b) Sketch the phase portrait in detail.

