1. Prove the following property of the matrix exponential:

$$e^{\mathbf{A}} \leq e^{\|\mathbf{A}\|}.$$

2. Consider the equation

$$\dot{x} = r^2 x^2 - 2x + r$$

for $r \in \mathbb{R}$ with r > 0.

- (a) Sketch the phase line, identify the equilibrium, and determine stability for representative values of r.
- (b) Draw the bifurcation diagram and identify the type of bifurcation as r varies.
- 3. A gradient system for $\boldsymbol{x} = [x_1, x_2, \dots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ is a system of the form

$$\boldsymbol{x}' = -\nabla V(\boldsymbol{x})$$
 or $x'_i = -\frac{\partial V}{\partial x_i}$ for $i = 1, 2, \dots, n$.

for some smooth function $V : \mathbb{R}^n \to \mathbb{R}$.

Suppose $\boldsymbol{x}(t)$ is a solution of the gradient system with initial condition $\boldsymbol{x}(0) = \boldsymbol{x_0}$.

- (a) Prove that $V(\boldsymbol{x}(t)) \leq V(\boldsymbol{x_0})$
- (b) Show that

$$x' = -x + 2y - x^3$$
 and $y' = 2x - y - y^3$

is a gradient system by finding an appropriate function V(x, y).

4. Assume that the function $u(x) \ge 0$ is of class $C[1, \infty)$, and

$$xu(x) \leq K + \int_{1}^{x} u(t) dt$$
 for $x \geq 1$.

Show that $u(x) \leq K$ for $x \geq 1$.

5. Consider the system

$$\begin{array}{rcl} x' &=& y-y^3\\ y' &=& -\left(x+y^2\right) \end{array}$$

- (a) Determine all equilibria and classify.
- (b) Sketch the phase portrait in detail.