UNIVERSITY OF Cincinnati

Department of Mathematical Sciences
PO Box 210025
Cincinnati OH 45221-0025

4^{th} Floor French Hall West
Phone (513) 556-4050
Fax (513) 556-3417

MATHEMATICS QUALIFYING EXAM, MAY 6, 2022

Four Hour Time Limit

In this exam $\mathbb{R}$ denotes the field of all real numbers, $\mathbb{R}^n$ is n-dimensional Euclidean space with the usual dot product $\vec{u}\cdot\vec{v} = \sum_{j=1}^m u_i v_i$, $\mathcal{M}_{m\times n}$ are m by n matrices with real entries. Proofs, or counter examples, are required for all problems.

1. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of real numbers such that for each integer $n \geq 2$ we have $|a_n - b_n| \leq 2^{-n}$. Prove that the limit $\lim_{n\to\infty} a_n$ exists if and only if the limit $\lim_{n\to\infty} b_n$ exists.

2. Prove that the series $\sum_{n=1}^{\infty} \frac{\sin(\pi n/2)}{n}$ is convergent. Also explain whether it is absolutely convergent.

3. For $x \in \mathbb{R}$, let

$$f_\alpha(x) := \begin{cases} |x|^\alpha \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) For what $\alpha > 0$ is f_α differentiable on $\mathbb{R}\setminus\{0\}$? Justify your answer.
(b) For what $\alpha > 0$ is f_α differentiable at 0? Justify your answer.

4. Let $h : (-1,1) \to \mathbb{R}$ be a differentiable function such that its derivative function h' is Riemann integrable on the interval $[-t,t]$ for each $0 < t < 1$. We define $H : (-1,1) \to \mathbb{R}$ by setting

$$H(x) = \begin{cases} \int_0^x h'(s)\,ds & \text{if } x > 0, \\ -\int_x^0 h'(s)\,ds & \text{if } x < 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that there is some real number C such that for each $x \in (-1,1)$ we have $H(x) = h(x) + C$.

5. Let $\mathcal{L}$ be the set of all 2×2 real matrices with trace zero, and let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(a) Show that $\mathcal{B} = \{H, E, F\}$ is a basis of $\mathcal{L}$.
(b) For $A, B \in \mathcal{L}$, denote

$$[A,B] = AB - BA.$$

Show that $X \mapsto [H,X]$ is linear on $\mathcal{L}$ and find its matrix expression in terms of $\mathcal{B}$.

6. Suppose that $A \in \mathcal{M}_{n\times n}$ is

$$A = \begin{bmatrix} 0 & 0 & 0 & \ldots 0 & 0 & -a_0 \\ 1 & 0 & 0 & \ldots 0 & 0 & -a_1 \\ 0 & 1 & 0 & \ldots 0 & 0 & -a_2 \\ 0 & 0 & 1 & \ldots 0 & 0 & -a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots 1 & 0 & -a_{n-2} \\ 0 & 0 & 0 & \ldots 0 & 1 & -a_{n-1} \end{bmatrix}$$

Prove that the characteristic polynomial is given by $\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$.

7. Let $\mathcal{C}$ be the space of all real continuous functions on $[-\pi,\pi]$.

(a) Verify that $\langle f,g\rangle := \int_{-\pi}^{\pi} f(x)g(x)dx$ defines an inner product on $\mathcal{C}$ with the corresponding norm $\|f\| = \sqrt{\langle f,f\rangle}$.
(b) Consider a three dimensional subspace $\mathcal{T}$ of $\mathcal{C}$ spanned by the functions $1, \sin x, \cos x$. Determine the function $t \in \mathcal{T}$ which is closest to the function

$$f(x) = \begin{cases} x & x \geq 0, \\ 0 & x < 0. \end{cases}$$

That is, use linear algebra to find function $t \in \mathcal{T}$ that achieves the minimum of $\|t - f\|^2$.

8. Let $B_1(0)$ denote the open unit ball in $\mathbb{R}^2$. Suppose $f : B_1(0) \to \mathbb{R}$ is continuously differentiable and that its gradient satisfies $\nabla f(x,y)\cdot(x,y) \geq 0$ for all $(x,y) \in B_1(0)$.

Prove that $f(0,0)$ is the absolute minimum of f on $B_1(0)$.