REAL ANALYSIS PRELIMINARY EXAM
MAY 2022

Time allowed: 2 hours 30 minutes.
Answer all problems and fully justify your work.
m is Lebesgue measure and χ_E the characteristic function of a set E.

Question 1

(a) Define Lebesgue outer measure on subsets of $\mathbb{R}$ and what it means for a subset of $\mathbb{R}$ to be Lebesgue measurable.

(b) Fix $0 < \alpha < 1$. Let F_1 be obtained from $[0,1]$ by removing the centered open interval of length $\alpha 3^{-1}$. Given $n \geq 2$ for which F_{n-1} has been defined and is a finite disjoint union of closed intervals, define F_n by removing the centered open interval of length $\alpha 3^{-n}$ from each of the intervals which form F_{n-1}. Let $F = \cap_{n=1}^{\infty} F_n$. Show that F is closed, $[0,1] \setminus F$ is dense in $[0,1]$, and $m(F) = 1 - \alpha$.

(c) Suppose $U \subset \mathbb{R}$ is a non-empty open set. Must it be true $m(U) > 0$? Give a proof or counterexample.

(d) Suppose $U \subset \mathbb{R}$ is a non-empty open set. Must it be true $m(\partial U) = 0$, where ∂U denotes the boundary of U? Give a proof or counterexample. **Hint:** Consider $U = F^c$ where F is defined in part (b).

Question 2

(a) Define what it means for a sequence of functions $f_n \colon D \to \mathbb{R}$ on a Lebesgue measurable set $D \subset \mathbb{R}$ to be Lebesgue measurable.

(b) Suppose $f_n \colon D \to \mathbb{R}$ is a sequence of Lebesgue measurable functions on a Lebesgue measurable set $D \subset \mathbb{R}$. Show that $\{x \in D : (f_n(x)) \text{ converges}\}$ is a Lebesgue measurable set.
Hint: Consider the Cauchy condition.

(c) Suppose $f_n \colon D \to \mathbb{R}$ is a sequence of Lebesgue measurable functions on a Lebesgue measurable set $D \subset \mathbb{R}$ that converge pointwise to a function $f \colon D \to \mathbb{R}$. Show that f is Lebesgue measurable.

(d) Suppose $f \colon \mathbb{R} \to [0, \infty)$ is a non-negative Lebesgue measurable function. Show that there exists an increasing sequence of simple functions $\varphi_n \colon \mathbb{R} \to [0, \infty)$ which converge pointwise to f. What is the relation between $\int_{\mathbb{R}} f \, \mathrm{d}m$ and $\int_{\mathbb{R}} \varphi_n \, \mathrm{d}m$?

Question 3

(a) State the monotone and dominated convergence theorems for a sequence of Lebesgue measurable functions $f_n \colon \mathbb{R} \to \mathbb{R}$, taking care to include all necessary hypotheses.

(b) Suppose $f_n \colon \mathbb{R} \to \mathbb{R}$ are a sequence of non-negative Lebesgue measurable functions which satisfy $f_{n+1}(x) \leq f_n(x)$ for every $x \in \mathbb{R}$ and converge pointwise to a function $f \colon \mathbb{R} \to \mathbb{R}$.

  (i) Show it need not be true that $\int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f$.

  (ii) Show that if we also assume f_1 is integrable, then $\int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f$.

(c) Let $f \colon \mathbb{R} \to \mathbb{R}$ be integrable with respect to Lebesgue measure. Show that $\lim_{n\to\infty} \int_{\mathbb{R}} |f|^{1/n} \, \mathrm{d}m$ exists and find the limit.
Hint: Let $A = \{x \in \mathbb{R} : |f(x)| > 0\}$, $B = \{x \in \mathbb{R} : |f(x)| \geq 1\}$, and consider integrals over B and $A \setminus B$ separately.

Question 4

(a) State Tonelli's theorem for a Lebesgue measurable function $f \colon \mathbb{R}^2 \to \mathbb{R}$, taking care to include all necessary hypotheses.

(b) Suppose $A, B \subset \mathbb{R}$ are Borel measurable. Show that the function $f \colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \chi_A(x + y)\chi_B(y)$ is Borel measurable.

(c) Given Borel sets $A, B \subset \mathbb{R}$, define $h \colon \mathbb{R} \to \mathbb{R}$ by $h(x) = m((A - x) \cap B)$, where $A - x := \{a - x : a \in A\}$. Show that h is Borel measurable and compute $\int_{\mathbb{R}} h(x) \, \mathrm{d}m(x)$.
Hint: First prove $\chi_E(y)\chi_F(y) = \chi_{E\cap F}(y)$ and $\chi_E(x+y) = \chi_{E-x}(y)$ for all $E, F \subset \mathbb{R}$ and $x, y \in \mathbb{R}$. Then consider $(x, y) \mapsto \chi_{A-x}(y)\chi_B(y)$.