# Statistical Methods Prelim Exam

1:00 pm - 3:30 pm, Thursday, May 5, 2022

UMVUE: uniformly minimum variance unbiased estimator

UMP: uniformly most powerful

1. Let $X_1, X_2, \ldots, X_n$ be a random sample of binary random variables with $P(X_i = 1) = p$, $i = 1, \ldots, n$ and $0 < p < 1$.
   (a) Show that UMVUE of $p(1-p)$ is $T_n = n\bar{X}(1-\bar{X})/(n-1)$.
   (b) Derive the asymptotic distribution of $T_n$.
   (c) Let $[p(1-p)]^{-1} I(0 < p < 1)$ be an improper prior density for $p$. Provided that the sample mean $\bar{X} \in (0,1)$, find the Bayes estimator of $p(1-p)$ under this improper prior and the squared error loss.
   (d) Discuss the bias and consistency of the Bayes estimator obtained in Part (c).

2. Consider the following density function

$$f(x, y; \sigma) = \frac{\sigma^2}{\pi\sqrt{3}} \exp\left\{-\frac{2\sigma^2}{3}[x^2 + y^2 - xy]\right\}$$

   for $(x, y) \in R^2$, where $\sigma(> 0)$ is the unknown parameter.
   (a) Suppose that $(X, Y)$ has the pdf $f(x, y; \sigma)$. Then, derive the UMP level $\alpha$ test for $H_0 : \sigma = 1$ versus $H_1 : \sigma > 1$ in its simplest implementable form.
   (b) Obtain explicitly the final expression of the power function for the test in Part (a) that does not involve any integral.

3. Let $X_1, \ldots, X_m$ be identically and independently distributed (i.i.d.) as the $\mathcal{N}(\mu_x, \sigma_x^2)$ distribution and let $Y_1, \ldots, Y_n$ be i.i.d. having the $\mathcal{N}(\mu_y, \sigma_y^2)$ distribution. Assume that $X_i$'s and $Y_j$'s are independent.
   (a) Assume that $\mu_x \in \mathcal{R}$, $\mu_y \in \mathcal{R}$, $\sigma_x^2 > 0$ , and $\sigma_y^2 > 0$. Find the UMVUE's of $\mu_x - \mu_y$ and $(\sigma_x/\sigma_y)^r, r > 0$.
   (b) Assume that $\mu_x \in \mathcal{R}$, $\mu_y \in \mathcal{R}$, $\sigma_x^2 = \sigma_y^2 = \sigma^2 > 0$. Find the UMVUE's of $\sigma^2$ and $(\mu_x - \mu_y)/\sigma$.

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d from a two-parameter exponential distribution $\mathcal{E}(a, b)$ with the density function as

$$f(x|a, b) = \frac{1}{b} \exp\left[-\frac{1}{b}(x - a)\right], \quad x > a, \quad b > 0.$$

   (a) Suppose that $b$ is known. Find a likelihood ratio test of size $\alpha$ for testing $H_0 : a = a_0$ versus $H_1 : a \neq a_0$.
   (b) Suppose both $b$ and $a$ are unknown. Find a likelihood ratio test of size $\alpha$ for testing $H_0 : b = b_0$ versus $H_1 : b \neq b_0$.

   **Hint: You might need the following results in Parts (a) and (b):**

   i. When $b$ is known, the distribution of $n[X_{(1)} - a]/b$ is the standard exponential distribution $\mathcal{E}(0, 1)$, where $X_{(1)}$ is the minimum order statistic.
   ii. When both $b$ and $a$ are unknown, $X_{(1)}$ and $\sum[X_i - X_{(1)}]$ are jointly sufficient and complete. They are independently distributed as

$$n[X_{(1)} - a]/b \sim \mathcal{E}(0, 1),$$
$$2\sum_{i=1}^{n}[X_i - X_{(1)}]/b \sim \chi^2_{2(n-1)}.$$