Statistics Prelim Exam, Stat Method part

10:00am-12:30pm, Thursday, August 22, 2024

r.s.: random sample; **i.i.d.**: identically and independently distributed;

c.s.s: complete sufficient statistics;

UMP: Uniformly Most Powerful; **UMVUE**: Uniformly Minimum Variance Unbiased Estimator;

1. Let $X_1, \ldots, X_n$ be a r.s. with the inverse Gaussian distribution, $IG(\lambda, \mu)$. Specifically, the density function is

$$f(x|\lambda, \mu) = \sqrt{\frac{\lambda}{2\pi}} \exp\left[(\lambda\mu)^{1/2}\right] x^{-3/2} \exp\left[-\frac{1}{2}\left(\frac{\lambda}{x} + \mu x\right)\right], \quad x > 0, \quad \lambda, \mu > 0.$$

   (a) Show that this density constitutes an exponential family.

   (b) Find the expectation of the inverse of X, that is, $\mathbb{E}(1/X)$.

   (c) Show that the statistics $\bar{X} = (1/n)\sum x_i$ and $S = \sum(1/x_i - 1/\bar{x})$ is jointly c.s.s. for (μ, λ).

   (d) Show that $\bar{X} \sim IG(n\lambda, n\mu)$ and $S \sim (1/\lambda)\chi^2_{n-1}$, where χ^2_{n-1} denote a chi-square distribution with df $n-1$.

2. Let $X_1, \ldots, X_m$ be i.i.d. having the $N(\mu_x, \sigma_x^2)$ distribution and let $Y_1, \ldots, Y_n$ be i.i.d. having the $N(\mu_y, \sigma_y^2)$ distribution. Assume that X_i's and Y_j's are independent and that $\mu_x \in \mathcal{R}$, $\mu_y \in \mathcal{R}$, $\sigma_x^2 > 0$, and $\sigma_y^2 > 0$. In your answers to the following questions, please clearly show all steps.

   (a) Find the UMVUE's of $\mu_x - \mu_y$ and $(\sigma_x/\sigma_y)^r$, $r > 0$ and is fixed.

   (b) Assume that $\mu_x = \mu_y (\in \mathcal{R})$ and that $\sigma_x^2/\sigma_y^2 = \gamma$ is known. Find the UMVUE of μ_x.

   (c) Assume that $\mu_x = \mu_y (\in \mathcal{R})$, $\sigma_x^2 > 0$, $\sigma_y^2 > 0$. Show that a UMVUE of μ_x does not exist.

3. Let $f(x, y; \sigma) = \frac{\sigma^2}{\pi\sqrt{3}} \exp\left\{-\frac{2\sigma^2}{3}[x^2 + y^2 - xy]\right\}$ for $(x, y) \in R^2$, where $\sigma (> 0)$ is the unknown parameter.

   (a) Suppose that (X, Y) has the pdf $f(x, y; \sigma)$. Then, derive the UMP level α test for for $H_0 : \sigma = 1$ versus $H_1 : \sigma > 1$ in its simplest implementable form. Obtain explicitly the final expression of the power function that does not involve any integral.

   (b) Suppose that $(X_1, Y_1), (X_2, Y_2)$ are i.i.d. with the common pdf $f(x, y; \sigma)$. Then, derive the UMP level α test for for $H_0 : \sigma = 1$ versus $H_1 : \sigma > 1$ in its simplest implementable form.

4. Let $\overline{X}$ be the sample mean of a random sample of size n from $N(\theta, \sigma^2)$ with a known $\sigma > 0$ and an unknown $\theta \in (-\infty, \infty)$. Let $\pi(\theta)$ be a prior density of θ.

   (a) Show that the posterior mean of θ, given $\overline{X} = x$, is of the form

$$E[\theta|\overline{X} = x] = x + \frac{\sigma^2}{n}\frac{d}{dx}\log(p(x)) ,$$

   where $p(x)$ is the marginal density of $\overline{X} = x$, unconditional on θ.

   (b) Show that the posterior variance of θ, given $\overline{X} = x$, is

$$Var[\theta|\overline{X} = x] = \frac{\sigma^2}{n} + \frac{\sigma^4}{n^2}\frac{d^2}{dx^2}\log(p(x)) .$$