Statistics Prelim Exam, Stat Method part

12:00 pm - 2:30 pm, Thursday, May 2, 2024

r.s.: random sample; **UMVUE**: Uniformly Minimum Variance Unbiased Estimator; **MLE**: Maximum Likelihood Estimator; **UMP**: Uniformly Most Powerful; **LRT**: Likelihood Ratio Test;

1. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a r.s. from $N(\mu, \sigma^2)$ with both $\mu \in (-\infty, \infty)$ and $\sigma^2 \in (0, \infty)$ **unknown** and $n \geq 2$.

   (a) Derive the MLE, say $\delta_1(\mathbf{X})$, and the UMVUE, say $\delta_2(\mathbf{X})$, of μ^2, respectively.

   (b) Derive the variance of $\delta_2(\mathbf{X})$.

   (c) For $i = 1, 2$, find the limiting distribution of $(\delta_i(\mathbf{X}) - \mu^2)$ **suitably normalized**.

   (d) Find the asymptotic relative efficiency (ARE) of $\delta_2(\mathbf{X})$ with respect to $\delta_1(\mathbf{X})$.

2. Suppose $X_1, \ldots, X_n$ is a r.s. from Geometric(p) distribution, where p is the probability of success and X is the number of trials needed before the first success, $0 < p < 1$ and $n \geq 1$. The probability mass function of X is $P(X = x) = p(1-p)^x, \quad x = 0, 1, 2, \ldots$.

   (a) Find the most powerful (MP) test of size $\alpha \in (0, 1)$ for $H_0 : p = p_0$ versus $H_1 : p = p_1$, $0 < p_1 < p_0 < 1$. In constructing the test, the exact sampling distribution of the test statistic should be clearly specified. Express your test in the simplest possible form and find the simplest expression for the constants to make the test size of α.

   (b) Derive the UMP test for $H_0 : p \geq p_0$ versus $H_1 : p < p_0, \quad 0 < p_0 < 1$. (Again, please clearly justify your answer.)

3. Let $X_1, \ldots, X_9$ and $Y_1, \ldots, Y_{12}$ represent two of independent r.s.'s from the respective normal distribution $N(\mu_1, \sigma^2)$ and $N(\mu_2, 3\sigma^2)$ where μ_1, μ_2 and σ^2 are unknown. Find the LRT for testing $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$ at the significance level α, $0 < \alpha < 1$.

   (Note: Please clearly specify the likelihood ratio, the exact distribution of the test statistic you use to achieve the size of α test and the explicit rejection region using the critical values from the specified sampling distribution of the test statistic.)

4. Let $X_1, \ldots, X_n$ be a r.s. from a population with its pdf $f(x|\theta) = e^{-(x-\theta)}$, $x > \theta$, $\theta > 0$. Assume that a prior distribution for θ is given by

   $$\pi(\theta) = \frac{1}{2} e^{-\frac{\theta}{2}} \ , \theta > 0 \ .$$

   (a) Find the posterior distribution of θ given $(X_1, \ldots, X_n)$, that is $\pi(\theta | X_1, \ldots, X_n)$.

   (b) Find the Bayes Rule under the squared error loss (SEL), $L(\theta, a) = (\theta - a)^2$.

   (c) Give the shortest 95% Bayesian interval estimate for θ.