Statistics and Probability Prelim Exam

9am-1pm, Wednesday, August 16, 2017

Note: To pass this exam, you need to pass both Statistics and Probability parts

Statistics Part

1. Suppose that $X_1, ..., X_n$ are iid from $Beta(\mu, 1)$ distribution, where $\mu > 0$.

   Suppose we want to test $H_0 : \mu \leq 1$ vs. $H_1 : \mu > 1$.

   (a) Find the likelihood ratio test statistic.

   (b) Is there a uniformly most powerful (UMP) test of size $\alpha = 0.05$? If no, give reasons for your answer. If yes, find the level $\alpha = 0.05$ UMP test specifying the critical region in a usable form.

2. Let $X_1, ..., X_n$ be i.i.d. from a population distribution with pdf

$$f(x|\theta) = e^{-(x-\theta)} \quad x > \theta, \;\; \theta > 0$$

   Suppose we assign a prior distribution for θ,

$$\pi(\theta) = \frac{1}{2} e^{-\frac{\theta}{2}} I(\theta > 0)$$

   a) Find the posterior distribution of θ, and the Bayes rule with respect to the quadratic loss $L(\theta, a) = (\theta - a)^2$.

   b) Give the shortest 95% Bayesian interval estimate for θ.

3. Let $X_1, ..., X_n$ be a random sample from $N(\mu_1, \sigma^2)$, and let $Y_1, ..., Y_n$ be a random sample from $N(\mu_2, \sigma^2)$, independent of the previous sample. Assume μ_1 and μ_2 are unknown, and σ is known.

   Find the UMVUE of $\eta = P(X < Y)$ where X and Y are independent with distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, respectively, which are the same as the ones given above.

4. A distribution often used to model income in the Pareto distribution with density,

$$f(x|\theta) = c^{1/\theta} \theta^{-1} x^{-(1+1/\theta)}, \quad , x > c$$

   where $\theta > 0$ and c is a known positive constant. Let $X_1, ..., X_n$ be a random sample from f.

   (a) If $\theta > 1$, find a method of moments estimate of θ using the first moment.

   (b) Compute its (asymptotic) efficiency with respect to the Cramer-Rao Lower Bound bound.

Probability Part

5. Let $\alpha > 0$ be a parameter to choose. Consider independent Binomial random variables $\{X_n\}_{n\in\mathbb{N}}$, each with parameter $(2, n^{-\alpha})$ respectively. That is,

$$\mathbb{P}(X_n = k) = \binom{2}{k} n^{-\alpha k}(1 - n^{-\alpha})^{2-k}, k = 0, 1, 2, n \in \mathbb{N}.$$

   So, $X_1, X_2, \dots$ form a random sequence of numbers $0, 1, 2$. We are interested in the pattern '012', the consecutive appearance of $0, 1, 2$: we say that such a pattern occurs at time t, if $X_{t-2} = 0, X_{t-1} = 1, X_t = 2$. Show that if $\alpha \in (1/3, \infty)$, with probability one there are only finitely many appearances of pattern '012' in the random sequence.

6. Let $\{X_n\}_{n\in\mathbb{N}}$ be a collection of independent random variables with

$$\mathbb{P}(X_n = \pm n^2) = \frac{1}{2n^\beta} \text{ and } \mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\beta}, n \in \mathbb{N},$$

   where $\beta \in (0, 1)$ is fixed for all $n \in \mathbb{N}$. Consider $S_n := X_1 + \cdots + X_n$. Show that

$$\frac{S_n}{n^\gamma} \Rightarrow \mathcal{N}(0, \sigma^2)$$

   for some $\sigma > 0, \gamma > 0$. Identify σ and γ as functions of β. You may use the formula

$$\sum_{k=1}^{n} k^\theta \sim \frac{n^{\theta+1}}{\theta + 1}$$

   for $\theta > 0$, and recall that by $a_n \sim b_n$ we mean $\lim_{n\to\infty} a_n/b_n = 1$.

7. Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with cumulative distribution function $\mathbb{P}(X_1 \leq x) = \left(\frac{x}{1+x}\right)^{1/2}$, $x \in [0, \infty)$. Show that

$$n^2 \left(\min_{i=1,\dots,n} X_i\right)$$

   converges weakly as $n \to \infty$. Identify the limiting distribution.

8. Prove that for any non-negative random variable X,

$$\mathbb{E}(\sin X) = \int_0^\infty \cos t \cdot \mathbb{P}(X > t) dt.$$