Statistics and Probability Prelim Exam

Wednesday, May 2nd, 2018, 12 pm - 4 pm

Note: To pass this exam, you need to pass both Statistics and Probability parts.

Statistics Part

1. Let $X_1, ..., X_n$ be a random sample from a population with pdf $N(0, \sigma^2)$, with $\sigma^2 > 0$ is unknown. Let $\theta = P(X > c)$ where X has the same $N(0, \sigma^2)$ distribution, and $c > 0$ is a known constant. Find the uniformly minimum variance unbiased estimator of θ.

2. Lett $X_1, ..., X_n$ be a random sample from a population with $Beta(\theta, 1)$ distribution, and let $Y_1, ..., Y_m$ be a random sample from a population with $Beta(\mu, 1)$ distribution, independent of the previous sample.

   Find the likelihood ratio test of size α for $H_0 : \theta = \mu$ vs. $H_1 : \theta \neq \mu$, giving the exact critical region in terms of the quantiles of a known distribution.

3. A Bayes estimator (or a Bayes rule) of a parameter θ based on a sample of size n is a decision rule $\delta(x_1, .., x_n)$ that minimizes the posterior expected loss.

   Let $\pi(\theta|x)$ be the posterior distribution.

   (a) For the loss function

   $$L(\theta, \delta) = (1 - \frac{\delta}{\theta})^2,$$

   find the general form of the Bayes estimator in terms of integral expressions involving the posterior distribution.

   (b) Suppose $X_i \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2), i = 1, \ldots, n$. The mean μ is assumed to be known and let the prior distribution of σ^2 be the inverse gamma distribution,

   $$f(\sigma^2|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-\alpha-1}e^{-\frac{\beta}{\sigma^2}},$$

   with $\alpha > 1, \beta > 0$ and mean $\beta/(\alpha - 1)$.
   (i) Give the exact expression of the pdf of the posterior distribution including its normalizing constant, and find the mean of the posterior distribution.

   (ii) Find the Bayes estimator of σ^2 under the loss function $L(\sigma^2, \delta) = (1 - \frac{\delta}{\sigma^2})^2$.

4. Let $X_1, ..., X_n$ be a sample from a population with the pdf

$$f(x;\theta) = \theta(\theta+1)x^{\theta-1}(1-x) \ ,$$

where $0 < x < 1$, $\theta > 0$.

(a) Show that $T_n = \frac{2\overline{X}}{1-\overline{X}}$ is a method of moments estimator of θ.

(b) Show that

$$\frac{\sqrt{n}(T_n - \mu_n(\theta))}{\sigma_n(\theta)} \to N(0,1)$$

in distribution where $\mu_n(\theta) = \theta$, $\sigma_n^2(\theta) = \theta(\theta+2)^2/2(\theta+3)$.

(c) Show taht T_n is not efficient by calculating the information bound.

Probability Part

5. Consider a collection of non-negative random variables $\{X_n\}_{n\in\mathbb{N}}$ on a common probability space. Suppose that

$$\mathbb{E}(X_n) = \frac{\sqrt{n}}{n+1}, \text{ for all } n \in \mathbb{N}.$$

   (a) Show that $X_n \to 0$ in probability as $n \to \infty$.

   (b) Find an explicit increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $X_{n_k} \to 0$ with probability one as $k \to \infty$.

6. Use the strong law of large numbers to prove that if $U_1, U_2, \ldots$ are i.i.d. uniform random variables on $(0, 1)$, then the sequence of geometric means converges with probability one as $n \to \infty$. That is,

$$\lim_{n\to\infty} (U_1 U_2 \cdots U_n)^{1/n} = c \text{ a.s.}$$

   Identify the limit c.

7. Suppose that $X_1, X_2, \ldots$ are independent random variables with distribution

$$\mathbb{P}(X_k = \pm 1) = \frac{1}{2k} \quad \text{and} \quad \mathbb{P}(X_k = 0) = \frac{1-k}{k}, \quad k \in \mathbb{N}.$$

   Prove that

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{\ln n}} \Rightarrow \mathcal{N}(0,1),$$

   as $n \to \infty$, where $\mathcal{N}(0, 1)$ is the standard normal distribution.

   Hints: You may use the following estimates

$$\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \int_{k}^{k+1} \frac{1}{k} dx \geq \sum_{k=1}^{n} \int_{k}^{k+1} \frac{1}{x} dx = \ln(n+1)$$

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \sum_{k=2}^{n} \int_{k-1}^{k} \frac{1}{k} dx \leq 1 + \sum_{k=2}^{n} \int_{k-1}^{k} \frac{1}{x} dx = 1 + \ln n.$$

8. Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. non-negative random variables with

$$\mathbb{P}(X_1 > x) = \frac{1}{(1+x)^2}, \quad x \geq 0.$$

   Prove that with an appropriately chosen parameter $\beta > 0$, we have that

$$\frac{1}{n^\beta} \max_{i=1,\ldots,n} X_i$$

   converges in distribution to a non-degenerate distribution. Identify in this case the value of β and the limit distribution (in term of the cumulative distribution function).