Statistics & Probability preliminary examination

August 20, 2014

Instructions

To pass this exam you need to pass both parts: probability and statistics.

Part I Statistics

#1. Let X_j, j = 1,2,...,n be independently distributed as $N(\alpha + j\beta, 1)$ where α and β are unknown. Find the **UMVU** estimator of β. Justify your answer.

#2. Let $X_1, ..., X_n$ be iid random variables with common density function $f(x) = \frac{1}{\pi}\frac{\lambda}{\lambda^2+x^2}$.

(2a) Show that the mle of λ, $\widehat{\lambda}_n$, exists,

(2b) Show that $\sqrt{n}(\widehat{\lambda}_n - \lambda) \xrightarrow{D} N(0, 2\lambda^2)$ as n $\to \infty$.

Hint : It is known that , for $a > -\frac{1}{2}$ and $b > a + \frac{1}{2}$,

$$\int_0^\infty \frac{(x^2)^a}{(1+x^2)^b}dx = \frac{1}{2}B(a+\tfrac{1}{2}, b-a-\tfrac{1}{2}) = \frac{\Gamma(a+\frac{1}{2})\Gamma(b-a-\frac{1}{2})}{2\Gamma(b)}.$$

#3. Let $X_1, ..., X_n$ be a random sample from Gamma(α, β) population, i.e. the common density function is $f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}$ for $x > 0$.

Show that there exist a UMP test for testing

$H_0 : \beta \leq \beta_0$ vs $H_1 : \beta > \beta_0$, α is known.

Write down the critical value for the level 5% UMP test .

#4. Suppose $X_1, ..., X_n$ are iid Poisson(μ) and $Y_1, ..., Y_n$ are iid Poisson(λ). Assume that $X_1, ..., X_n$ are independent of $Y_1, ..., Y_n$. Suppose the prior density for (μ, λ) is

$\pi(\mu, \lambda) = e^{-(\mu+\lambda)}$, for $\mu, \lambda > 0$.

Suppose that we observe data for which $\sum_{j=1}^n X_j = 10$ and $\sum_{j=1}^n Y_j = 20$.

[**4a**] Find the joint posterior distribution of (μ, λ) and marginal posterior distribution of

$$\theta = \frac{\mu}{\mu + \lambda}.$$

[**4b**] Give the Bayes estimator of θ under the square error loss.

You may use known results as long as you state them clearly.

Part II Probability

#5. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with

$$P(X_i = 1) = P(X_i = -1) = 1/2.$$

Prove that

$$\frac{\sqrt{3}}{\sqrt{n^3}} \sum_{k=1}^{n} k X_k \Rightarrow N(0,1)$$

(You may use formulas $\sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1)$ and $\sum_{j=1}^{n} j^3 = \frac{1}{4} n^2 (n+1)^2$ without proof.)

#6. Let (X, Y) be a pair of strictly positive random variables with cumulative distribution function $F(x, y)$. Show that

$$E\left(\frac{1}{XY}\right) = \iint_{\mathbb{R}^2} \frac{F(x,y)}{x^2 y^2} dx dy$$

#7. Let $A_1, A_2, \ldots$ be a sequence of independent events. Let $\mathcal{F}_n$ be the sigma algebra generated by $A_n, A_{n+1}, \ldots$. Define

$$\mathcal{T} = \bigcap_{n \geq 1} \mathcal{F}_n.$$

Let $B \in \mathcal{T}$. Show that B has probability 0 or 1.

#8. Suppose that $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ are independent random variables such that

$$P(X_k = k) = P(X_k = -k) = \frac{1}{2k^\alpha}, \; P(X_k = 0) = 1 - \frac{1}{k^\alpha},$$

$$P(Y_k > x) = e^{-kx} \text{ for } x > 0.$$

Show that if $\alpha > 1$ then the series $\sum_{k=1}^{\infty} X_k Y_k$ converges with probability one.