Statistics and Probability Preliminary Examination
August 2015

Note: To pass this exam, you need to pass both Statistics and Probability parts.

PART I - STATISTICS

1. Let $X_1,\ X_2,\ \ldots\ ,\ X_n$ be a random sample from the inverse Gaussian distribution $I(\mu, \tau)$ with probability density

$$\sqrt{\frac{\tau}{2\pi x^3}\exp\left(-\frac{\tau}{2x\mu^2}(x-\mu)^2\right)}\,, x > 0, \tau, \mu > 0.$$

   (a) Show that $V = \frac{\tau}{\mu^2 X}(X-\mu)^2$ has χ^2_1 distribution.

   (b) Find the uniformly most powerful test with significance level $\alpha = 0.05$ for $H_0 : \mu \leq \ \mu_0$ against $H_1 : \mu > \mu_0$ when τ is known and μ_0 is given.

2. Let $X_1,\ \ldots,\ X_n$ be iid having the common pdf

$$\frac{1}{\sigma}\exp\{-\frac{(x-\mu)}{\sigma}\}I(x > \ \mu)$$

   where μ is unknown but σ is known with $-\infty < \mu < \infty$, and $0 < \sigma < \infty$.

   (a) Show that $U = X_{(1)}$ is a complete sufficient statistic for μ, and find the UMVUE of μ.

   (b) Find the conditional expectation, $E\left[\ \overline{X}\ \middle|\ U\ \right]$.

3. Let $X_1,\ \ldots,\ X_n$ be iid having the common pdf

$$\frac{1}{\sigma}\exp\{-\frac{(x-\mu)}{\sigma}\}I(x > \ \mu)$$

   where μ is unknown but σ is known with $-\infty < \mu < \infty$, and $0 < \sigma < \infty$. Assume a prior distribution for μ given by

$$\pi(\mu) = e^{-\mu},\ \mu > 0.$$

   Find the Bayes estimate for μ using quadratic loss function. (Hint: Note the range of values of μ.)

4. Let $X_1, ...X_n$ be iid with pdf,

$$f(x|\theta) = c^{1/\theta}\theta^{-1}x^{-(1+1/\theta)} \qquad x > c$$

where $c > 0$ is known, and $0 < \theta < 1$ is unknown.

(i) Find the method of moment estimator, $T(X_1, ..., X_n)$, of θ using the first moment.
(ii) Find asymptotic efficiency of $T(X_1, ..., X_n)$ with respect to the lower bound for the variance of unbiased estimators of θ.

PART II - PROBABILITY

5. Let $\{U_n\}_{n\in\mathbb{N}}$ be a collection of i.i.d. random variables with $\mathbb{E}U_n = 0$ and $\mathbb{E}U_n^2 = \sigma^2 \in (0, \infty)$. Consider random variables $\{X_n\}_{n\in\mathbb{N}}$ defined by $X_n = U_n + U_{2n}, n \in \mathbb{N}$, and the partial sum $S_n = X_1 + \cdots + X_n$. Find appropriate constants $\{a_n, b_n\}_{n\in\mathbb{N}}$ such that

$$\frac{S_n - b_n}{a_n} \Rightarrow \mathcal{N}(0, 1).$$

6. State the Kolmogorov's 3 series theorem. Prove the sufficient part.

7. Consider random variables X and $\{X_n\}_{n\in\mathbb{N}}$ defined on certain common probability space.

   (i) Provide an example such that X_n converges to X in distribution, but not in probability.

   (ii) Suppose that X_n converges to X in probability. Prove that X_n converges to X in distribution.

8. Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with mean μ, variance $\sigma^2 \in (0, \infty)$ and finite fourth moment. Consider $Y_n = X_nX_{n+1}, n \in \mathbb{N}$. Prove that

$$\frac{Y_1 + \cdots + Y_n}{n} \to \mu_Y \text{ in probability}$$

as $n \to \infty$ for some constant μ_Y, and identify μ_Y.