Statistics and Probability Preliminary Examination

May 2015

Note: To pass this exam, you need to pass both Statistics and Probability parts.

PART I - STATISTICS

1. Let $X_1, X_2, ..., X_n$ be iid according to the uniform distribution $U(\alpha - \beta, \alpha+\beta)$, where α and β are both unknown. Find the UMVU estimators of α and β.

2. Let $X_1, ..., X_{n_1}$ and $Y_1, ..., Y_{n_2}$ represent two independent random samples from the respective normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$. Here, all parameters are unknown but it is given that $\sigma_1^2 = 4\sigma_2^2$. Give the exact Likelihood Ratio Test for testing $H_0 : \mu_1 = \mu_2$ v.s. $H_a : \mu_1 \neq \mu_2$ at significance level α, stating the rejection region in terms of the percentile of a familiar distribution.

3. Let $X_1, ..., X_n$ be iid random sample with $Beta(\theta, 2)$ distribution with pdf
$$f(x; \theta) = \theta(\theta + 1)x^{\theta-1}(1 - x) \quad 0 < x < 1, \theta > 0.$$
   (a) Show that the estimator $T_n = \frac{2\bar{X}}{1-\bar{X}}$ is asymptotically normally distributed in the sense $\sqrt{n}(T_n - \theta)$ converges in distribution to $N(0, \sigma^2(\theta))$ for some function $\sigma^2(\theta)$, and give the expressions for $\sigma^2(\theta)$.
   (b) Determine if T_n is asymptotically efficient in the sense that it asymptotically achieves the lower bound for variance of unbiased estimator of θ.

4. Let $X_1, ..., X_n$ be iid random sample from $U(\theta, \theta + 1)$, where $-\infty < \theta < \infty$ and it is unknown. Assume a prior distribution for θ given by the probability density function
$$\pi(\theta) = \frac{1}{2}e^{-|\theta|}, \quad -\infty < \theta < \infty.$$
Find the Bayes estimate of θ with respect to the quadratic loss function, i.e., find the posterior mean.

PART II - PROBABILITY

5. Let $\{U_n\}_{n\in\mathbb{N}}$ be a collection of i.i.d. random variables distributed uniformly on interval $(0,1)$. Consider a triangular array of random variables $\{X_{n,k}\}_{k=1,\ldots,n,n\in\mathbb{N}}$ defined as

$$X_{n,k} = \mathbf{1}_{\{\sqrt{n}U_k\le 1\}} - \frac{1}{\sqrt{n}}.$$

Find constants $\{a_n, b_n\}_{n\in\mathbb{N}}$ such that

$$\frac{X_{n,1}+\cdots+X_{n,n}-b_n}{a_n} \Rightarrow \mathcal{N}(0,1).$$

6. Prove the following weak law of large numbers for triangular arrays. Let $\{X_{n,k}\}_{k=1,\ldots,n,n\in\mathbb{N}}$ be independent random variables *without any assumptions on the moments*. Let $Y_{n,k} = X_{n,k}1_{\{|X_{n,k}|\le n\}}$. Suppose that

   (i) $\lim_{n\to\infty}\sum_{k=1}^n \mathbb{P}(|X_{n,k}|>n) = 0$ and

   (ii) $\lim_{n\to\infty} n^{-2}\sum_{k=1}^n \mathbb{E}(Y_{n,k}^2) = 0$.

Set $S_n = X_{n,1}+\cdots+X_{n,n}$ and $a_n = \sum_{k=1}^n \mathbb{E}Y_{n,k}$. Show that

$$\frac{S_n-a_n}{n} \to 0 \text{ in probability as } n\to\infty.$$

7. Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with $\mathbb{E}|X_i|<\infty$. Show that

$$\lim_{n\to\infty}\frac{\max_{i=1,\ldots,n}X_i}{n} = 0 \text{ almost surely.}$$

Hint: You may want to first consider X_n/n.

8. Let $X_1, X_2, \ldots$ be independent random variables with

$$\mathbb{P}(X_n=1)=\mathbb{P}(X_n=-1)=\frac{1}{2n}, \mathbb{P}(X_n=0)=1-\frac{1}{n}.$$

Let $Y_0=0$ and for $n\ge 1$, set

$$Y_n = \begin{cases} X_n & \text{if } Y_{n-1}=0 \\ nY_{n-1}|X_n| & \text{if } Y_{n-1}\ne 0. \end{cases}$$

Show that Y_n converges to zero in probability, but not almost surely.