Statistics and Probability Prelim Exam

9am-1pm, Wednesday, May 4, 2016

Note: To pass this exam, you need to pass both Statistics and Probability parts

Statistics Part

1. Suppose $X_1, ..., X_n$ are iid with pdf $f(x|\theta) = \theta e^{-\theta x}$, $x > 0$, where $\theta > 0$ is unknown. Assume a prior distribution given by $\pi(\theta) = e^{-\theta}$, $\theta > 0$.

   (a) Find the posterior distribution of θ and the posterior mean.

   (b) Now, assume a loss function for estimating θ given by $L(\theta, a) = e^{(a-\theta)} - (a - \theta) - 1$. Find the Bayes rule for estimating θ based on this loss function.

2. Let $X_1, ..., X_n$ be a random sample from $Poisson(\mu)$ distribution. Suppose we want to estimate $\theta = P(X_1 = 0) = e^{-\mu}$. Let T_1 be the maximum likelihood estimator of θ, and let $T_2 = (1/n)\sum_i I_{\{0\}}(X_i)$ where $I_{\{0\}}(x)$ is the indicator function which is equal to 1 if $x = 0$, and is equal to 0 otherwise. Note that T_2 is the proportion of zeroes in the sample.

   (a) Find T_1 and show that T_1 and T_2 are each asymptotically normally distributed.

   (b) Find the asymptotic relative efficiency of T_2 with respect to T_1.

   (c) Determine if either or both of T_1 and T_2 are unbiased for θ.

3. Suppose that $X_1, \ldots, X_n$ are iid Poisson(λ) where $0 < \lambda < \infty$ is unknown.

   (a) Show that $T(X_1, \ldots, X_n) = \sum_{i=1}^n X_i$ is a complete sufficient statistic for λ.

   (b) Find the Uniformly Minimum Variance Unbaised Estimator (UMVUE) of $\tau_1(\lambda) = \lambda^2$.

   (c) Find the UMVUE of $\tau_2(\lambda) = \lambda e^{-\lambda}$, that is $P(X_1 = 1) = \lambda e^{-\lambda}$.

4. Let $X_1, \ldots, X_n$ be a sample from the *inverse Gaussian* distribution $I(\mu, \tau)$ with density

$$\sqrt{\frac{\tau}{2\pi x^3}} \exp\left(-\frac{\tau}{2x\mu^2}(x-\mu)^2\right), \quad x > 0, \ \ \tau, \mu > 0.$$

   (a) Find the moment generating function of X_1.

   (b) Show that $V = \frac{\tau}{X\mu^2}(X - \mu)^2 \sim \chi_1^2$

   (c) Show that $\overline{X} = \sum_{i=1}^n X_i/n \sim I(\mu, n\tau)$.

   (d) Show that there exists a UMP test for testing $H_0 : \mu \leq \mu_0$ versus. $H_1 : \mu > \mu_0$ when τ is known.

   (e) Show that there exists a UMP test for testing $H_0 : \tau \leq \tau_0$ versus. $H_1 : \tau > \tau_0$ when μ is known.

Probability Part

5. Prove the following identity: for all $p \geq 1$ and any random variable X,

$$\mathbb{E}|X|^p = p \int_0^\infty x^{p-1} \mathbb{P}(|X| > x) dx.$$

6. Consider i.i.d. Poisson random variables $\{X_n\}_{n\in\mathbb{N}}$ with parameter $\lambda > 0$.
Consider $Y_n := X_n X_{2n}, n \in \mathbb{N}$ and $T_n := Y_1 + \cdots + Y_n$.

   (a) Compute $\mathbb{E}T_n$.

   (b) Find an explicit constant C such that $\mathrm{Var}(T_n) \leq Cn$ for all $n \in \mathbb{N}$. Explain clearly how you determine the constant C. The constant does not have to be optimal and may depend on λ. However, it must be independent from n.

   (c) Find a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$, such that

$$\frac{T_n}{a_n} \to 1 \text{ in probability.}$$

   Justify your result.

7. Let $X_1, X_2, \ldots$ be independent non-negative random variables. Suppose that there exists constants $K < \infty$ such that $\sup_{n\in\mathbb{N}} \mathbb{E}\exp(X_n) \leq K$.

   (a) Show that for all $x > 0, n \in \mathbb{N}$,

$$\mathbb{P}(X_n > x) \leq Ke^{-x}.$$

   (b) Show that for all $\epsilon > 0$,

$$\mathbb{P}\left(\frac{X_n}{\log n} > 1 + \epsilon \text{ i.o.}\right) = 0.$$

   You may use the inequality in part (a) directly.

8. (a) State Lindeberg–Feller central limit theorem.

   (b) Use Lindeberg–Feller central limit theorem to prove the following. Given a sequence $\{X_n\}_{n\in\mathbb{N}}$ of independent but *not necessarily identically distributed* random variables with mean zero, variance 2 and $\mathbb{E}|X_k|^4 < 17$, show that the central limit theorem for the partial sums $S_n := X_1 + \cdots + X_n$ holds in the form of

$$\frac{S_n - b_n}{a_n} \Rightarrow \mathcal{N}(0, 1),$$

   and specify the normalizing constants $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$.