April 2014 Prelim - Statistics Part:

Problem 1

Let $X_1, X_2, ..., X_n$ be i.i.d from $N(\mu, \sigma^2)$ population. Suppose that $\sigma^2 = \lambda\mu^2$ with unknown $\lambda > 0$ and $\mu \in \mathbb{R}$. Find a Likelihood Ratio test for testing

$$H_0 : \lambda = 1 \quad \text{vs.} \quad H_1 : \lambda \neq 1.$$

Problem 2

Suppose that $\{X_1, X_2, ..., X_n\}$ and $\{Y_1, Y_2, ..., Y_n\}$ are independent samples, X_i's from $\mathcal{E}(\lambda)$ population, and Y_j's from $\mathcal{E}(\mu)$ population; λ and μ are unknown positive numbers . (The density function of X_1 is: $f(x) = \frac{1}{\lambda} e^{-x/\lambda}$, $x > 0$.)

[2a] Find the UMVUE W of $\theta = \frac{\mu}{\lambda}$.

[2b] Show that $\frac{n}{n-1}\frac{\mu}{\lambda} W$ follows F- distribution.

Problem 3

Let $X_1, X_2, ..., X_n$ be i.i.d. from Gamma distribution Gamma(k, θ) with an unknown $\theta > 0$, and known k. Let the prior be such that $\varpi = \frac{1}{\theta}$ has the Gamma distribution $\text{Gamma}(\alpha, \gamma)$ with known $\alpha >$ 0 and $\gamma > 0$.

{ The density function of Gamma distribution $\text{Gamma}(\alpha, \beta)$ is : $f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$, $x > 0$ }

[3a] Under the square error loss, find the Bayes estimator of θ, $T(X_1, X_2, ..., X_n)$.

[3b] Compute the bias of $T(X_1, X_2, ..., X_n)$ as an estimator of θ.

Problem 4

If $X_1, X_2, ..., X_n$ are iid U(0, θ) random variables, it is known that the UMVE of

$$\theta \text{ is } \quad \delta_n = \frac{n+1}{n} X_{[n;n]} = \frac{n+1}{n} \bullet \max\{X_1, X_2, ..., X_n\}$$

[4a] Determine the limit distribution of $n(\theta - \delta_n)$ as $n \to \infty$

[4b] Find the limit $\lim_{n\to\infty} \frac{E[X_{[n,n]} - \theta]^2}{E[\delta_n - \theta]^2}$

Part II Probability

Problem 5. Let $\{X_{nk} : k = 1, \ldots, n, n \in \mathbb{N}\}$ be a family of independent random variables satisfying

$$P\left(X_{nk} = \frac{k}{\sqrt{n}}\right) = P\left(X_{nk} = -\frac{k}{\sqrt{n}}\right) = P(X_{nk} = 0) = 1/3$$

Let $S_n = X_{n1} + \cdots + X_{nn}$. Prove that S_n/s_n converges in distribution to a standard normal random variable for a suitable sequence of real numbers s_n.

Some useful identities:

$$\begin{aligned} \sum_{k=1}^{n} k &= \frac{1}{2}n(n+1) \\ \sum_{k=1}^{n} k^2 &= \frac{1}{6}n(n+1)(2n+1) \\ \sum_{k=1}^{n} k^3 &= \frac{1}{4}n^2(n+1)^2 \end{aligned}$$

Problem 6. Suppose $\{X_n\}$ is a sequence of independent nonnegative random variables.

Show that if $\sum_{n=1}^{\infty} E(\min\{X_n, 1\}) < \infty$ then $\sum_n X_n < \infty$ with probability one.

Problem 7. Let $X_1, X_2, ...$ be independent and identically distributed random variables with $E|X_1| = \infty$.

(a) Show that for each real number $a > 0$

$$\sum_{n=1}^{\infty} P(|X_n| \geq an) = \infty.$$

(b) Show that

$$\sup_n \frac{1}{n}|X_n| = \infty$$

with probability one.

(c) Show that

$$\sup_n \frac{1}{n}|X_1 + X_2 + ... + X_n| = \infty$$

with probability one.

Problem 8. (a) Let X be a positive random variable. Show by a method of your choice that

$$E(X) = \int_0^{\infty} P(X > x)dx.$$

(b) Explain why

$$\int_0^{\infty} P(X > x)dx = \int_0^{\infty} P(X \geq x)dx.$$