Statistics & Probability preliminary examination

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Instructions To pass this exam you need to pass both parts: probability and statistics.

Part I Statistics

1. (Stat) Let $X_1, X_2, ..., X_n$ be a random sample from the Cauchy location family, i.e. the common density function is:

$$f_\theta(x) = \frac{1}{\pi\{1 + (x-\theta)^2\}}$$

where $\theta \in \mathbb{R}$. Let $M_n = \text{median}(X_1, X_2, ..., X_n)$ be the sample median. Compute the relative efficiency of $\{M_n$: n = 1,2,...$\}$ as a sequence of estimators of the location parameter θ.

Hint: You may use the following well-known result about Order Statistics:

Theorem 1 *Suppose that $X_1, X_2, ..., X_n$ is a random sample from a population with continuous distribution F, density function f. Let G be the inverse function of F. Assume that λ is a positive number such that $f(G(\lambda)) > 0$, and $\{\alpha_n\}$ is a sequence of positive integers such that* $\lim_{n\to\infty} \sqrt{n}\{\frac{\alpha_n}{n} - \lambda\} = 0$.

Then

$$\sqrt{n}\left\{X_{[n;\alpha_n]} - G(\lambda)\right\} \xrightarrow{D} N\left(0, \frac{\lambda(1-\lambda)}{\{f(G(\lambda))\}^2}\right) \text{ as } n \to \infty.$$

where $X_{[n;\alpha_n]}$ is the α_nth smallest among $X_1, X_2, ..., X_n$.

2. (Stat). Let $X_1, X_2, ..., X_n$ be a random sample from a population with density function

$$f(x|\theta) = \theta c^\theta x^{-(\theta+1)},\ x > c,$$

where $c > 0$ is known. Use a prior

$$\lambda(\theta) = \frac{b^a \theta^{a-1} e^{-b\theta}}{\Gamma(a)},\ \theta > 0,$$

where $a > 0$, $b > 0$ are both user-specified and known. Find the posterior distribution and the Bayes estimator (under the square error loss), i.e. the posterior mean, of θ.

3. (Stat) Let $X_1, X_2, ..., X_n$ be a random sample from the Poisson distribution with mean $\theta > 0$.
   (a) Find the (uniformly) minimum variance unbiased estimator of $a\theta + be^{-\theta} + c$, where a, b, and c are given positive numbers.
   (b) Does there exist an unbiased estimator of $\frac{1}{\theta}$? JUSTIFY your answer!

4. (Stat) Assume that $X_1, X_2, ..., X_n$ is a random sample from the Rayleigh distribution with density function

$$f(x;\theta) = \frac{2x}{\theta}\exp\left(-\frac{x^2}{\theta}\right),\ x > 0,$$

where $\theta > 0$ is an unknown parameter. Obtain the Likelihood Ratio Test (LRT) statistic for testing

$H_0 : \theta = \theta_0$ versus $H_0 : \theta \neq \theta_0$

where $\theta_0 > 0$ is a given constant. Write down the α- level critical region for the LRT as an inequality in terms of a sufficient statistic, and show how you would find the critical region that is valid for any sample size.

Part II Probability

5. (Prob) Suppose $X_{n,1}, X_{n,2}, ...$ are independent random variables centered at expectations (mean 0) and set $s_n^2 = \sum_{k=1}^n E(X_{n,k})^2$. Assume for all k that $|X_{n,k}| \le M_n$ with probability 1 and that $M_n/s_n \to 0$. Let $Y_{n,i} = 3X_{n,i} + X_{n,i+1}$. Show that
$$\frac{Y_{n,1} + Y_{n,2} + ... + Y_{n,n}}{s_n}$$
converges in distribution and find the limiting distribution.

6. (Prob) Let $X_1, X_2, ...$ be independent and identically distributed random variables with distribution function
$$F(x) = \begin{cases} 0 & \text{if } x \le 1 \\ 1 - x^{-a} & \text{if } x \ge 1 \end{cases}$$
where $a > 0$. Show that $Z_n = n^{-1/a} \max(X_1, X_2, ..., X_n)$ converges in distribution and find its limit.

7. (Prob) Let $X_1, X_2, \ldots$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$. Suppose X_n has mean μ_n and variance σ_n^2. Assume that $\lim_{n\to\infty} \mu_n = \mu \in \mathbb{R}$ and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$. Show that $X_n \to \mu$ almost surely.

8. (Prob) Prove Kolmogorov's inequality: Let $X_1, \ldots, X_n$ be independent with mean zero and finite variances, and let $S_k = \sum_{j=1}^k X_j$. Then, for $t > 0$, prove that
$$\Pr(\max_{1\le k\le n} |S_k| \ge t) \le \frac{\mathrm{Var}(S_n)}{t^2}$$