Statistics & Probability preliminary examination

12 – 4 PM, August 21, 2013

Instructions

To pass this exam you need to pass both parts: probability and statistics.

Part I Statistics

Problem 1. Let X_i have exponential distribution with mean λ/θ_i, for $i = 1, .., n$. Let Y_i have exponential distribution with mean $\lambda\theta_i$, for $i = 1, .., n$. Assume that $X_1, ..., X_n, Y_1, ..., Y_n$ are all independent.

The parameters $\theta_1, ..., \theta_n$ and λ are all unknown positive numbers.

Find the maximum Likelihood estimate (MLE), $\hat{\lambda}$, of λ,

Also find the bias and variance of $\hat{\lambda}$.

Problem 2. Let $X_1, X_2, ..., X_n$ be a random sample from Beta(μ,1) and $Y_1, Y_2, ..., Y_n$ a random sample from Beta(θ,1) distribution, and both samples are independent. Find a likelihood ratio test of

$H_0 : \mu = \theta$ vs $H_1 : \mu \neq \theta$.

Express the test statistic in terms of

$$T = \frac{\sum_{j=1}^{n} \log(X_j)}{\sum_{j=1}^{n}\{\log(X_j) + \log(Y_j)\}}$$

and use it to find a test of size $\alpha = 0.05$, that is valid for any sample size

Problem 3. Let $X_1, X_2, ..., X_n$ be i.i.d. from Beta(θ, θ) distribution with an unknown $\theta > 0$. Let the prior of θ be $\mathcal{E}(\lambda)$ distribution with known λ , i. e. the prior density is $\frac{1}{\lambda}e^{-\frac{\theta}{\lambda}}$, $\theta > 0$. Under the square error loss, find the Bayes estimate (posterior mean) and the posterior standard deviation of θ.

Problem 4. Suppose that $\{X_1, X_2, ..., X_n\}$ and $\{Y_1, Y_2, ..., Y_n\}$ are independent samples, X_i's from $\mathcal{E}(\lambda)$ population, and $Y_j's$ from $\mathcal{E}(\mu)$ population; λ and μ are unknown positive numbers.

Find the UMVUE (Uniformly Minimum Variance Unbiased Estimator) of $\theta = P(X_1 > Y_1)$.

Part II Probability

Problem 5. Suppose that for each $n = 1, 2, \dots$ we have $P(X_n \leq Y_n \leq Z_n) = 1$. Assume that X_n converges to Z in probability, and Z_n converges to the same random variable Z in probability as $n \to \infty$. Show that Y_n converges to Z in probability as n $\to \infty$.

Problem 6. Suppose $X_1, Y_1, X_2, Y_2, \dots,$ are independent identically distributed with mean zero and variance 1. For integer n, let

$$U_n = \frac{1}{n}\left(\sum_{j=1}^{n} X_j\right)^2 + \frac{1}{n}\left(\sum_{j=1}^{n} Y_j\right)^2.$$

Prove that $\lim_{n\to\infty} P(U_n \leq u) = 1 - e^{-u/2}$ for $u > 0$.

Problem 7. Prove Etemadi's inequality: Let $X_1, \dots, X_n$ be independent, and denote $S_k = \sum_{j=1}^{k} X_j$. Then for $t \geq 0$

$$P(\max_{1\leq k\leq n} |S_k| \geq 3t) \leq 3 \max_{1\leq k\leq n} P(|S_k| \geq t)$$

Problem 8. Suppose $X > 0$ has cumulative distribution function $F(x)$. Prove that

$$E\left(\frac{1}{1+X}\right) = \int_0^\infty \frac{F(t)}{(1+t)^2} dt$$