Sample Questions for the PhD Preliminary Exam in Statistics and Probability

Department of Mathematical Sciences
University of Cincinnati
January 2013

Statistics

1. Suppose that $X_1, \ldots, X_n$ are i.i.d. Poisson(θ), and θ has the Gamma(a, b) prior distribution with mean ab, show that, given $(X_1, \ldots, X_n) = (x_1, \ldots, x_n)$, the posterior distribution is Gamma(a + $\sum_{j=1}^n x_j$, $\frac{b}{1+nb}$).

2. Let X and Y be independent exponential random variables, with densities $f_\lambda(x) = \frac{1}{\lambda}e^{-\frac{x}{\lambda}}[[x > 0]]$ and $g_\mu(y) = \frac{1}{\mu}e^{-\frac{y}{\mu}}[[y > 0]]$, respectively. We observe Z and W with

   $Z = \min(X, Y)$ and $W = \begin{cases} 1 & if \quad Z = X \\ 0 & if \quad Z = Y \end{cases}$.

   Find the MLE of λ and μ.

3. Suppose that X $\sim$ Beta (α, β), show that the Fisher information matrix is:

   $$I(\alpha, \beta) = \begin{pmatrix} \psi'(\alpha) - \psi'(\alpha+\beta) & -\psi'(\alpha+\beta) \\ -\psi'(\alpha+\beta) & \psi'(\beta) - \psi'(\alpha+\beta) \end{pmatrix},$$

   where $\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$ (the digamma function) and $\psi'(x) = \frac{d}{dx}\psi(x)$ (the trigamma function.)

4. Let X_1, X_2,...,X_n denote the incomes of n person chosen at random from a certain population. Suppose that

   $$f(x, \theta) = c^\theta \theta x^{-(1+\theta)}, \ x > c,$$

   where $\theta > 1$ and $c > 0$. (c is known)

   a Express mean income μ in terms of θ.

   b Find the optimal test statistic for testing H : $\mu = \mu_0$ vs. K : $\mu > \mu_0$

   c Use the central limit theorem to find a normal approximation to critical value of test in part [b].

5. Suppose that $X_1, ..., X_n$ are i.i.d. Poisson(θ), and θ has the Gamma(a, b) prior distribution whose mean is ab, show that, for $a > k$, the Bayes estimator under the loss function $L_k(d, \theta) = \frac{(\theta-d)^2}{\theta^k}$ is given by

$$\delta_k(\overline{x}) = \frac{E(\Theta^{1-k}|\overline{x})}{E(\Theta^{-k}|\overline{x}))} = \frac{b}{1+nb}(n\overline{x} + a - k),$$

where $\overline{x} := \frac{1}{n}\sum_{j=1}^{n} x_j$.

6. Suppose that $X_1,\ X_2, \ldots, X_n$ is a random sample from N(μ, σ^2) with unknown $\mu \in \mathbb{R}$ and unknown $\sigma^2 > 0$. Let $\theta = (\theta_1, \theta_2) = (\mu, \sigma^2) \in \Theta = \mathbb{R} \times \mathbb{R}_+$.

   (a) Find the UMVUE $\widehat{q}_n(X_1, X_2, ..., X_n)$ of $q(\theta) = \frac{\theta_1}{\sqrt{\theta_2}} = \frac{\mu}{\sigma}$.

   (b) Find the limiting distribution of $\sqrt{n}\{\widehat{q}_n(X_1, X_2, ..., X_n) - \frac{\mu}{\sigma}\}$ as $n \to \infty$.

Probability

1. Suppose $X_1, X_2, \ldots$ are independent identically distributed random variables taking values $1, \ldots, r$ with probabilities $p_1, \ldots, p_r$. Let $L(X_1, \ldots, X_n)$ denote the likelihood of the sequence $X_1, \ldots, X_n$. Prove that

$$-\frac{1}{n}\log L(X_1, \ldots, X_n) \to h$$

where $h = -\sum_{k=1}^{r} p_r \log p_r$ is Shannon's entropy.

2. Suppose that $X_1 \le X_2 \le \ldots$ and that $X_n \xrightarrow{P} X$. Show that $X_n \to X$ with probability 1.

3. Suppose X_n has density $f_n(x) = 1 + \cos(2\pi n x)$ on interval $[0, 1]$. Show that $X_n \Rightarrow X$ for some X.

4. Suppose $X_1, X_2, \ldots$ are independent random variables with distribution $\Pr(X_k = 1) = p_k$ and $\Pr(X_k = 0) = 1 - p_k$. Prove that if $\sum Var(X_k) = \infty$ then

$$\frac{\sum_{k=1}^{n}(X_k - p_k)}{\sqrt{\sum_{k=1}^{n} p_k(1-p_k)}} \Rightarrow N(0, 1).$$

5. Prove that for $X > 0$

$$E(e^X) = 1 + \int_0^\infty e^t \Pr(X > t)\, dt.$$

6. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Show that $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ imply $f(X_n, Y_n) \xrightarrow{P} f(X, Y)$.