Statistics and Probability Prelim Exam

Tuesday, August 21, 2018 12pm - 4 pm

Note: To pass this exam, you need to pass both Statistics and Probability parts Statistics Part

Statistics Part

1. Suppose that $\{X_1, .., X_n\}$ is a random sample from an exponential distribution with an unknown parameter $\beta > 0$ and pdf given by

$$f(x|\beta) = \frac{1}{\beta}\exp\{-x/\beta\}; \quad x > 0,$$

   and that $\{Y_1, ..., Y_n\}$ is random sample from an exponential distribution with an unknown parameter $\delta > 0$. Assume that the two samples are independent. Let $\theta = P(X_1 < Y_1)$. Find the uniformly minimum variance unbiased estimator of θ based on the random samples given above when $n = 2$.

2. Suppose $X_1, ..., X_n$ are iid from a distribution with pdf

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)} & \text{if } x > \theta \\ 0 & else, \end{cases}$$

   where $\theta > 0$ is unknown. We want to test the null hypothesis $H_0 : \theta \leq \theta_0$ versus the alternative hypothesis $H_1 : \theta > \theta_0$ for a known value $\theta_0 > 0$. Is there a uniformly most powerful (UMP) test ? If there is one, find the UMP test and give the critical region of size $\alpha = 0.05$. If there is none, explain why.

3. Let $X_1, ...X_n$ be iid $Bernoulli(p)$, $0 < p < 1$. Let T_n be the uniformly minimum variance unbiased estimator of p^2. Find T_n and determine if it is asymptotically normally distributed in the sense that $\sqrt{n}(T_n - \mu)$ converges in distribution to a normal distribution, for some constant μ.

4. Let $x_1, ..., x_n$ be a random sample from $Bernoulli(\theta)$ distribution, where $0 < \theta < 1$, and is unknown. Assume a $U(0, 1)$ prior for θ. Consider the loss function for estimating θ given by

$$L(\theta, a) = \frac{(\theta - a)^2}{(1 - \theta)}.$$

   Find the Bayes rule with respect to the above loss function.

Probability Part

5. Let X be a non-negative random variable. Prove that

$$\mathbb{E}e^X = 1 + \int_0^\infty e^t \mathbb{P}(X > t)dt.$$

6. Consider a sequence of independent random variables $\{X_n\}_{n\in\mathbb{N}}$, each X_n has the following distribution:

$$\mathbb{P}(X_n = k) = \frac{1}{(n+5)^{k\gamma}}, k = 1, 2, 3,$$

and $\mathbb{P}(X_n = 0) = 1 - \mathbb{P}(X_n \in \{1, 2, 3\})$, for some parameter $\gamma > 0$. So, each realization of $\{X_n\}_{n\in\mathbb{N}}$ is an infinite sequence of $0, 1, 2, 3$. Find the range of γ so that *both* of the following conditions are satisfied at the same time.

   (a) The number 1 occurs infinitely often with probability one.

   (b) The number 3 occurs only a finite number of times with probability one.

   Justify your answer.

7. Suppose that $X_1, X_2, \ldots$ are independent random variables, and X_n is uniformly distributed over $[0, n]$ (i.e., $\mathbb{P}(X_n \le x) = x/n, x \in [0, n], n \in \mathbb{N}$). Find appropriate sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ so that

$$\frac{\sum_{k=1}^n X_k - b_n}{a_n} \Rightarrow \mathcal{N}(0, 1),$$

as $n \to \infty$, where $\mathcal{N}(0, 1)$ is the standard normal distribution. Justify your answer. *Hints:* You may use the following estimates

$$\sum_{k=1}^n k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

8. Suppose that for each $n \in \mathbb{N}$, random variable X_n has density $f_n(x) = 1 + \cos(2\pi n x)$ on $[0, 1]$. Prove that X_n converges in distribution to some random variable X as $n \to \infty$, and determine the law of X.