Name:____________________ M#:________ Instructor:________________

Instructions: This exam has seven problems on seven pages. Show all your work, expressing yourself in a clear and concise manner. Do as many problems as you can, but be advised that a complete solution to a problem may be worth more than several partial ones. Use the backs of the exam pages for work, if necessary. No calculators of any kind are allowed.

1. Find the derivative of $f(x) = |x|^3$ (the answer should be written as a single formula).

Solution: Since $|x| = \sqrt{x^2}$ you can write $f(x) = |x|^3 = (x^2)^{3/2}$. The chain rule says that

$$f'(x) = \frac{3}{2}(x^2)^{\frac{1}{2}} \cdot 2x.$$

So $f'(x) = 3x|x|$.

2. Let $f(x)$ be an infinitely differentiable periodic function of period 2π so that $f(x+2\pi) = f(x)$ for every x. Show that

$$\int_0^{2\pi} f'''(x)f(x)\,dx = 0\,.$$

Solution: The periodicity of the function implies the periodicity of its derivatives as shown by differentiating the equation $f(x+2\pi) = f(x)$ as many times as we like.

Integrate by parts using the periodicity of the function to eliminate boundary terms.

$$\int_0^{2\pi} f'''(x)f(x)\,dx = -\int_0^{2\pi} f''(x)f'(x)\,dx = -\frac{1}{2}f'^2(x)\Big|_0^{2\pi} = 0\,.$$

3.

$$\int_0^1 \frac{x}{x\sqrt{x}+1} dx = ?$$

Hint: use substitution and partial fractions. Note that $t = -1$ is a root of the polynomial $1 + t^3$.

Solution: Set $u = \sqrt{x}$ and then integrate $2u^3/(u^3+1)$ by partial fractions.

To begin, set $u = \sqrt{x}$ so that $\frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u}$. Then u goes from 0 to 1 as x does.

Since $\frac{u^3}{u^3+1} = 1 - \frac{1}{(u+1)(u^2-u+1)}$ we write

$$\frac{1}{(u+1)(u^2-u+1)} = \frac{A}{u+1} + \frac{Bu+C}{u^2-u+1}$$

and require

$$\begin{aligned} A + B &= 0 \\ B + C - A &= 0 \\ A + C &= 1 \end{aligned}$$

which says we want $A = 1/3$, $B = -1/3$ and $C = 2/3$.

$$\begin{aligned} 2\int_0^1 \frac{u^3}{u^3+1}\, du &= 2\int_0^1 1 - \left[\frac{1/3}{u+1} + \frac{(-1/3)u + 2/3}{u^2-u+1}\right] du \\ &= 2\left(u - \frac{1}{3}\log(u+1)\right)\Bigg|_0^1 + \int_0^1 \frac{2}{3}\frac{u-2}{u^2-u+1}\, du \\ &= 2(1 - \log(2)/3) + \frac{2}{3}\int_0^1 \frac{u-2}{u^2-u+1}\, du \end{aligned}$$

In the remaining integral, write $v = u^2 - u + 1$ so that $dv = 2u - 1$ and

$$\begin{aligned} \int \frac{u-2}{u^2-u+1}\, du &= \frac{1}{2}\int \frac{2u-4}{u^2-u+1}\, du \\ &= \frac{1}{2}\int \frac{dv}{v} - \frac{3}{2}\int \frac{1}{u^2-u+1}\, du \\ &= \frac{1}{2}\log(u^2-u+1) - \frac{3}{2}\int \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}}. \end{aligned}$$

The remaining antiderivative is an arctan. In all we get the the antiderivative

$$\frac{1}{3}\left(6\sqrt{x} - 2\log(\sqrt{x}+1) + \log(x - \sqrt{x} + 1) - 2\sqrt{3}\arctan(\frac{2}{\sqrt{3}}(\sqrt{x} - \frac{1}{2}))\right).$$

4. Assume that $a > -1$ and $b > -1$. Calculate the limit

$$\lim_{n\to\infty} n^{b-a} \frac{1^a + 2^a + \cdots + n^a}{1^b + 2^b + \cdots + n^b}.$$

Hint: Look for Riemann sums in the expression.

Solution: The expression in question can be written as

$$\frac{\left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \cdots + \left(\frac{n}{n}\right)^a}{\left(\frac{1}{n}\right)^b + \left(\frac{2}{n}\right)^b + \cdots + \left(\frac{n}{n}\right)^b}.$$

Multiplying top and bottom by $1/n$ we see that we have right Riemann sums for $\int_0^1 x^a\,dx$ and $\int_0^1 x^b\,dx$ both of which are convergent (if they're improper) because $a, b > -1$. So the limit of the ratio is the ratio of the limits, namely

$$\frac{\int_0^1 x^a\,dx}{\int_0^1 x^b\,dx} = \frac{b+1}{a+1}.$$

5. (a) Show that the infinite series $\sum_{n=0}^{\infty} ne^{-n}$ converges.

   (b) Compute the sum of the series from part (a). Hint: What is the sum $\sum x^n$ and how is it related to the sum $\sum nx^n$?

Solution: For $|x| < 1$ we have that the geometric series, $\sum_0^{\infty} x^n = \frac{1}{1-x}$, is absolutely convergent. This tells us that, on the same interval, $\sum nx^{n-1} = \frac{1}{(1-x)^2}$ and so, $\sum_0^{\infty} nx^n = \frac{x}{(1-x)^2}$.

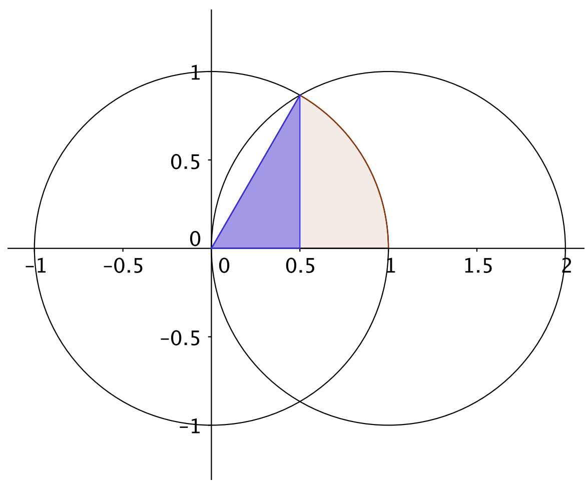
So $\sum_{n=0}^{\infty} ne^{-n} = \left.\frac{x}{(1-x)^2}\right|_{x=1/e}$

6. The centers of two disks of radius 1 are one unit apart. Find the area of the union of the two disks.

Solution:

The area of the overlap of the two disks is made up of 4 congruent regions each of which has the area of a sector (1/6 of one disk) less half the area of an equilateral triangle with side length 1.

$$
\begin{aligned}
|D_1 \cup D_2| &= |D_1| + |D_2| - |D_1 \cap D_2| \\
&= 2\pi - 4(\pi/6 - \frac{1}{4} \cdot 1 \cdot \frac{\sqrt{3}}{2}).
\end{aligned}
$$



If you *really* want to do an integral, you can. The region making 1/4 the overlap of the disks (shown in the figure) can be described in polar coordinates as $\sec(\theta)/2 < r < 1$ for $0 < \theta < \pi/3$ so its area can be written as

$$(1/2) \int_a^b [r_2^2 - r_1^2] \, d\theta = \frac{1}{2} \int_0^{\pi/3} 1 - \frac{1}{4} \sec^2(\theta) \, d\theta.$$

Then the hardest thing is to recall that the derivative of tan is $\sec^2$.

7. Compute the limit as $t \to \infty$ of the average value of the function $f(x) = (1 - 2/x)^x$ on the interval $[2, t]$. Show that any rules or laws you use do, in fact, apply.

Solution:

We must evaluate

$$\lim_{x\to\infty} \frac{1}{x-2} \int_2^x (1-2/t)^t \, dt.$$

Since (using L'Hospital for the ratio has the form 0/0 in line 2 below)

$$\begin{aligned}
\lim_{x\to\infty} (1-2/x)^x &= \exp\left(\lim_{x\to\infty} x \log(1-2/x)\right) \\
&= \exp\left(\lim_{x\to\infty} \frac{\log(1-2/x)}{1/x}\right) \\
&= \exp\left(\lim \frac{(2/x^2)\frac{1}{1-2/x}}{-1/x^2}\right) \\
&= \exp\left(-2 \lim \frac{1}{1-2/x}\right) \\
&= \exp(-2),
\end{aligned}$$

we see that $\int_2^\infty f(x)\, dx$ diverges because the integrand has non-zero limit. Therefore we may apply L'Hospital to the limit of the average value which has form ∞/∞.

We find, using the fundamental theorem of calculus, that

$$\lim_{x\to\infty} \frac{\int_2^x (1-2/t)^t \, dt}{x-2} = \lim_{x\to\infty} (1-2/x)^x = e^{-2}.$$