Name:________________________ M#:__________ Instructor:____________________

Instructions: This exam has seven problems on seven pages. Show all your work, expressing yourself in a clear and concise manner. Do as many problems as you can, but be advised that a complete solution to a problem may be worth more than several partial ones. Use the backs of the exam pages for work, if necessary. No calculators of any kind are allowed.

1. Find $\frac{d}{dx}(f(x))$ if $\frac{d}{dx}(f(3x)) = 6x$.

Solution: Answer: $f'(x) = 2x/3$.

Set $u = 3x$. Then $f'(u) = \frac{2}{3}u$, so $f'(x) = \frac{2}{3}x$.

2. Find a polynomial $f(x)$ with the property that $f(-1)$ and $f(1)$ are local maxima and $f(0) = -1$ is a local minimum.

Solution: Require that $f'(-1) = f'(0) = f'(1)$, that $f' > 0$ for $x < -1$ and $0 < x < 1$ while $f' < 0$ for $-1 < x < 0$ and $x > 1$.

One such function is $f'(x) = -x(x+1)(x-1)$. In that case $f(x) = x^2/2 - x^4/4$ has the required extremal properties and $f(x) = x^2/2 - x^4/4 - 1$ also has $f(0) = -1$.

3. What is the maximum value of $g(x) = |\sin(x) - 2\cos(x)|^2$?

Solution: Answer: 5.

Pick θ in $(0, \pi/2)$ so that $\cos(\theta) = 1/\sqrt{5}$. Then $\sin(\theta) = 2/\sqrt{5}$ and

$$\begin{aligned}\sin(x) - 2\cos(x) &= \sqrt{5}(\sin(x)\cos(\theta) - \cos(x)\sin(\theta)) \\ &= \sqrt{5}\sin(x-\theta)\end{aligned}$$

Therefore

$$g(x) = 5\sin^2(x-\theta)$$

from which we see that the maximum value of g is 5 since $\sin^2$ assumes its maximum value 1.

4. For what values of the constant t does the function

$$F(x) = (x^2 + t)e^x$$

have two distinct inflection points?

Solution: Answer: $t < 2$.

$F'' = (x^2 + 4x + 2 + t)e^x$ the requirement that F'' have two distinct roots reduces to $16 - 4(2 + t) > 0$ and that says $t < 2$.

5. Evaluate

$$\int_0^1 \sqrt[8]{1 - x^6} - \sqrt[6]{1 - x^8}\, dx$$

Hint: what is the inverse of the function $y = (1 - x^a)^b$?

Solution: Answer:

$$\int_0^1 \sqrt[8]{1 - x^6} - \sqrt[6]{1 - x^8}\, dx = 0.$$

The functions $\sqrt[6]{1 - x^8}$ and $\sqrt[8]{1 - x^6}$ are both monotone decreasing on $0 \leq x \leq 1$ and each is the inverse of the other. That means their graphs are reflections of each other in the line $y = x$ as are the areas under the graph of one of them and to the left of the graph of the other. These area are represented by $\int_0^1 \sqrt[6]{1 - x^8}\, dx$ and $\int_0^1 \sqrt[8]{1 - x^6}\, dx$. Reflected areas are congruent. So the integral over $[0, 1]$ of the difference of the functions is 0.

You *could* try to actually evaluate the antiderivative and that'd be a fine, if more time consuming, approach. And, you'd need to know facts about hypergeometric functions. Perhaps easier is the following: start by evaluating

$\int_0^1 \sqrt[6]{1 - x^8}\, dx$ by substitution setting $y = y(x) = \sqrt[6]{1 - x^8}$. Then $y(0) = 1$ and $y(1) = 0$ and $x(y) = \sqrt[8]{1 - y^6}$ so (integrating by substitution and then integrating by parts)

$$\begin{aligned}
\int_0^1 \sqrt[6]{1 - x^8}\, dx &= \int_1^0 y(x)x'(y)\, dy \\
&= yx(y)\Big|_1^0 - \int_1^0 x(y)\, dy \\
&= 0 - \int_1^0 x(y)\, dy \\
&= \int_0^1 \sqrt[8]{1 - y^6}\, dy
\end{aligned}$$

6. For which values of t is the series

$$\sum_{n\geq 1} n^{1/n^t} - 1$$

convergent?

Solution: Answer: The series converges if and only if $t > 1$.

Write the sum as $\sum_{n\geq 1} e^{\frac{\log(n)}{n^t}} - 1$.

Since $\lim_{x\to 0} \frac{e^x-1}{x} = 1$ the limit comparison test says it is sufficient to consider the series

$$\sum_{n\geq 1} \frac{\log n}{n^t}.$$

If $t \leq 1$ then $\log(n)/n^t > 1/n$ for large n and the series diverges by comparing with the harmonic series.

It $t > 1$ then for large n we have $\log(n) < n^{\frac{t-1}{2}}$ so that for large n,

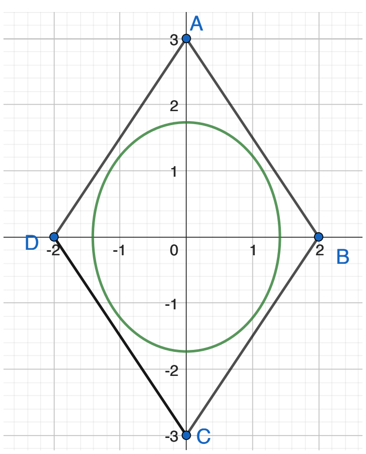
$$\frac{\log n}{n^t} < \frac{1}{n^{(t+1)/2}}.$$

And then, because $(t+1)/2 > 1$, the series

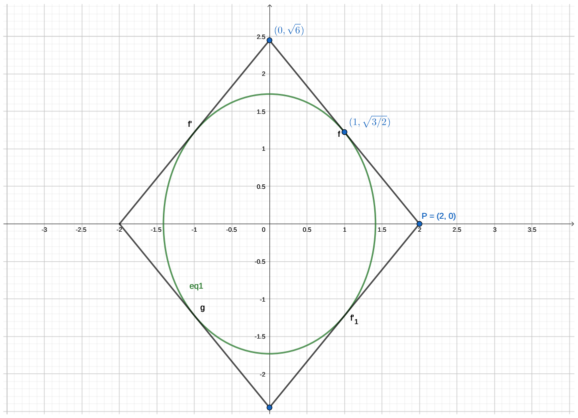
$$\sum_{1} \frac{1}{n^{(t+1)/2}}$$

converges.

7. Of all the parallelograms $ABCD$ having A and C on the y-axis, having B and D on the x-axis and containing the ellipse $x^2/2 + y^2/3 = 1$, which has the smallest area?



Solution: Answer: the one with vertices $(\pm 2, 0)$, $(0, \pm\sqrt{6})$ and area $A = 4\sqrt{6}$.



Due to the symmetry of the ellipse with respect to the x, y, axes it is enough to work in the first quadrant and minimize the area of a right triangle with 90 angle at the origin. Clearly the triangle is minimized when the hypotenuse is tangent to the ellipse.

Setting

$$u = x/\sqrt{2}$$
$$v = y/\sqrt{3}$$

gives the ellipse the equation $u^2 + v^2 = 1$ and changes areas by

$$\Delta x \Delta y = \sqrt{6} \Delta u \Delta v.$$

At the point $(u, v) = (\cos\theta, \sin\theta)$ the tangent to the circle has slope $-\cot\theta$ and equation

$$v - \sin\theta = -\cot\theta(u - \cos\theta).$$

The u and v intercepts of this line are $(\frac{1}{\cos\theta}, 0)$ and $(0, \frac{1}{\sin\theta})$ respectively so the area of the triangle is

$$A = \frac{1}{2}\frac{1}{\sin\theta}\frac{1}{\cos\theta} = \frac{1}{\sin 2\theta}.$$

The area is minimized when $\sin 2\theta$ takes its maximal value — at $\theta = \pi/4$ and the minimal value is $A = 1$. The point of tangency is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

In the x, y plane the intercepts are $x = \sqrt{2}u = 2$ and $y = \sqrt{3}v = \sqrt{6}$, and the minimal area is then $A = \sqrt{6}$.

So the whole parallelogram we needed to find has vertices $(\pm 2, 0)$, $(0, \pm\sqrt{6})$ and area $A = 4\sqrt{6}$.