Name:____________________ M#:__________ Instructor:____________________

Instructions: This exam has seven problems on seven pages. Show all your work, expressing yourself in a clear and concise manner. Do as many problems as you can, but be advised that a complete solution to a problem may be worth more than several partial ones. Use the backs of the exam pages for work, if necessary. No calculators of any kind are allowed.

1. Proof or counter-example: If f is an odd, differentiable function defined for all x then f' is an even function.

Solution: f is odd exactly when $-f(-x) = f(x)$ for all x. Differentiating, (using the chain rule!) this says

$$f'(-x) = f'(x)$$

which says f' is an even function.

2. Evaluate $\int_0^{\pi/2} \sqrt{\sin(x)+1}\,dx$

Solution: One approach is to write $\sin x = \sin 2(x/2) = 2\sin(x/2)\cos(x/2)$ so that

$$\begin{aligned} 1+\sin x &= \cos^2(x/2)+\sin^2(x/2)+2\sin(x/2)\cos(x/2) \\ &= (\cos(x/2)+\sin(x/2))^2 \end{aligned}$$

Since $\cos(x/2)+\sin(x/2) \geq 0$ on the domain of integration

$$\sqrt{(\cos(x/2)+\sin(x/2))^2} = |\cos(x/2)+\sin(x/2)| = \cos(x/2)+\sin(x/2),$$

so the integral becomes

3. Find a function $f(x)$ differentiable for all $x \neq 0$ that satisfies

   1. $f'(x) = \frac{-10}{x^2}$ for $x \neq 0$
   2. $f(-2) = 3$
   3. $f(1) = -2$

Solution: Finding an antiderivative isn't hard. The point is that different constants of integration can be used on different intervals of continuity. So this works:

$$f(x) = \begin{cases} \frac{10}{x} + 8 & \text{if } x < 0 \\ \frac{10}{x} - 12 & \text{if } x > 0 \end{cases}$$

4. Find all values of k for which the limit exists:

$$\lim_{x\to 3} \frac{4x^2 + kx + 7k - 6}{2x^2 - 5x - 3}.$$

List all possible values of k, find the limit for each such k, and explain why no other k's exist.

Solution: Unless the polynomial in the numerator of this rational function vanishes at $x = 3$ the limit of the numerator is nonzero and of the denominator is 0 as $x \to 3$ — so the entire limit doesn't exist. So the only values of k for which the limit can exist are those for which

$$4(3^2) + 3k + 7k - 6 = 0,$$

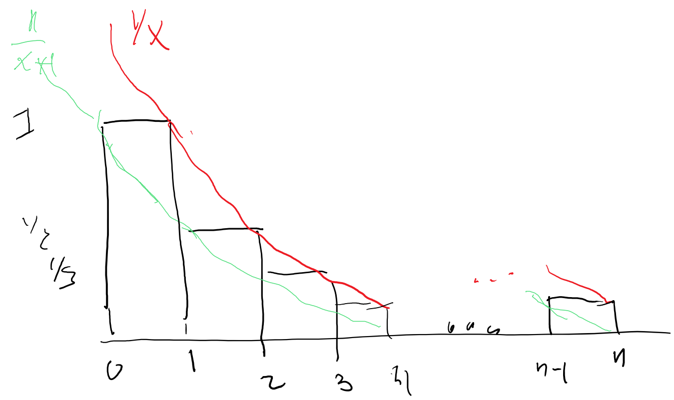
That is, $k = -3$. In this case

$$\begin{aligned} \lim_{x\to 3} \frac{4x^2 - 3x - 27}{2x^2 - 5x - 3} &= \lim_{x\to 3} \frac{(x-3)(4x+9)}{(x-3)(2x+1)} \\ &= \lim_{x\to 3} \frac{(4x+9)}{(2x+1)} \\ &= 21/7 = 3. \end{aligned}$$

5. Find the limit or prove that it doesn't exist:

$$\lim_{N\to\infty} \frac{\sum_{1\le n\le N} \frac{1}{n}}{\log(N)}$$

Solution: The limit exists and equals 1.



Comparing the functions $1/(x+1)$ and $1/x$ with the step function taking the values $1/n$ on the intervals $[n-1, n)$ for $n = 1, 2, 3, \cdots$ we see that

$$\int_0^n \frac{1}{x+1}\,dx \le \sum_{1\le k\le n} \frac{1}{k} \le 1 + \int_1^n \frac{1}{x}\,dx$$

so that

$$\log(n+1) \le \sum_{1\le k\le n} \frac{1}{k} \le 1 + \log(n).$$

Therefore

$$\frac{\log(n+1)}{\log(n)} \le \frac{\sum_{1\le k\le n} \frac{1}{k}}{\log(n)} \le 1 + \frac{1}{\log n}.$$

The expressions on the left and right in this inequality both have limit 1 as $n \to \infty$.

6. Let $w(x)$ be a positive-valued function on the interval $[a, b]$, and suppose that $f(x)$ is a continuous function on $[a, b]$. Define the *weighted-average* u of $f(x)$ over the interval $[a, b]$ by
$$u = \frac{\displaystyle\int_a^b w(x)f(x)\,dx}{\displaystyle\int_a^b w(x)\,dx}.$$
Show that there exists c in (a, b) satisfying $f(c) = u$.

   Hint: f is continuous on $[a, b]$ so it attains a minimum value m and a maximum value M on $[a, b]$, satisfying
$$m \leq f(x) \leq M.$$
   Start with multiplying this inequality by an appropriate function and remember to use the Intermediate Value Theorem when the time comes!

Solution: The intermediate value theorem assures us that for every y with $m \leq y \leq M$ there is a c between a and b so that $f(c) = y$.

Since $w > 0$ we can calculate as follows

$$\begin{gathered}
m \leq f(x) \leq M \\
w(x)m \leq w(x)f(x) \leq w(x)M \\
m\int_a^b w(x)\,dx \leq \int_a^b w(x)f(x)\,dx \leq M\int_a^b w(x)\,dx \\
m \leq \frac{\int_a^b w(x)f(x)\,dx}{\int_a^b w(x)\,dx} \leq M
\end{gathered}$$

And so there is a c between a and b (inclusive) for which

$$f(c) = \frac{\int_a^b w(x)f(x)\,dx}{\int_a^b w(x)\,dx}.$$

7. Compute the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \arctan(k/n)$$

Solution: Notice that the sums in question are Riemann sums with uniform partitions for a definite integral with integrand $\arctan(x)$. So the limit is $\int_0^1 \arctan(x)\, dx$. You can evaluate this by finding an antiderivative with one application of integration by parts.