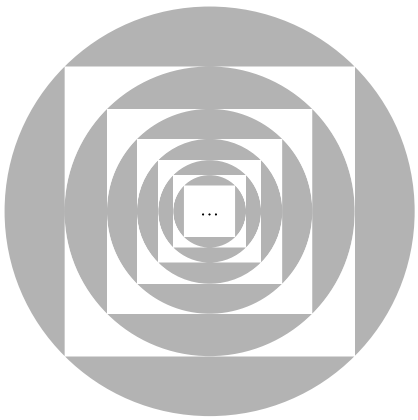
Solutions Calculus Contest 2011

1. A square S_1 is inscribed in the unit circle C_1 center at the origin. Inside the square S_1 a second circle C_2 with center the origin is inscribed and a second square S_2 is inscribed in the second circle. This is repeated infinitely many times. Find the sum of the areas $\sum(\text{Area } C_i - \text{Area } S_i)$ pictured as the gray regions in the diagram below



SOLUTION. The area of a circle of radius r minus the area of an inscribed square is $\pi\, r^2 - \left(\frac{2\,r}{\sqrt{2}}\right)^2 = r^2\,(\pi - 2)$ since the diagonal of the square is the diameter of the circle which means that the inscribed square has side $\frac{2\,r}{\sqrt{2}}$. The circle inscribed in the square has radius $\frac{r}{\sqrt{2}}$. This means that the sum of the areas between the circle and inscribed squares is

$$\text{Areas} = (\pi - 2)\left(1^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \ldots\right) = (\pi - 2)\,\frac{1}{1 - 1/2} = 2\,(\pi - 2) = 2.28319.$$

2. Suppose that f is a continuous real-valued function on $[0, 2]$ such that $f(0) = f(2)$. Show that there a number ζ with $1 \le \zeta \le 2$ such that $f(\zeta) = f(\zeta - 1)$.

SOLUTION. The function $g(\zeta) = f(\zeta) - f(\zeta - 1)$ has the property

$$g(1) = f(1) - f(0) \qquad \text{and}$$

$$g(2) = f(2) - f(1) = f(0) - f(1).$$

This means that g changes sign on $[1, 2]$. Due to the Intermediate Value Theorm, there is a ζ in $[1, 2]$ with $g(\zeta) = 0$ or equivalently, with $f(\zeta) = f(\zeta - 1)$.

3. Evaluate $\int \frac{x+1}{x^2+\sqrt{2}\,x+1}\,dx$.

SOLUTION. We have that

$$\int \frac{x+1}{x^2+\sqrt{2}\,x+1}\,dx = \frac{1}{2}\int \frac{2\,x+\sqrt{2}}{x^2+\sqrt{2}\,x+1}\,dx + \frac{1}{2}\int \frac{2-\sqrt{2}}{x^2+\sqrt{2}\,x+1}\,dx$$

$$= \frac{1}{2}\ln\left|x^2 + \sqrt{2}\,x + 1\right| + \frac{2-\sqrt{2}}{2}\int \frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\,dx$$

$$\left(\frac{u}{\sqrt{2}} = \left(x + \frac{1}{\sqrt{2}}\right)\right.$$

$$= \frac{1}{2}\ln\left|x^2 + \sqrt{2}\,x + 1\right| + \frac{2-\sqrt{2}}{2}\int \frac{1}{\frac{u^2}{2}+\frac{1}{2}}\,\frac{du}{\sqrt{2}}$$

$$= \frac{1}{2}\ln\left|x^2 + \sqrt{2}\,x + 1\right| + \frac{2-\sqrt{2}}{\sqrt{2}}\int \frac{1}{u^2+1}\,du$$

$$= \frac{1}{2}\ln\left|x^2 + \sqrt{2}\,x + 1\right| + \frac{2-\sqrt{2}}{\sqrt{2}}\arctan\left(\sqrt{2}\left(x + \frac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{1}{2}\ln\left|x^2 + \sqrt{2}\,x + 1\right| + \frac{2-\sqrt{2}}{\sqrt{2}}\arctan\left(\sqrt{2}\,x + 1\right)\Big)$$

4. A rectangle has two sides along the positive coordinate axes and its upper right hand corner point lies on the curve $x^3 - 2xy^2 + y^3 + 1 = 0$. How fast is the area of the rectangle changing as the point passes through passes through the position $(2, 3)$ if it is moving so that $\frac{dx}{dt} = 1$ units per second.

SOLUTION. Taking the derivative of $x^3 - 2xy^2 + y^3 + 1 = 0$, we get

$$3x^2\dot{x} - 2\dot{x}y^2 - 4xy\dot{y} + 3y^2\dot{y} = 0.$$

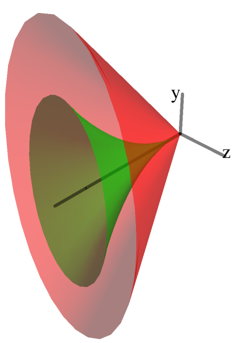
Substituting $x = 2,\ y = 3,\ \dot{x} = 1$, we get

$$12 - 18 - 24\dot{y} + 27\dot{y} = 0$$

which gives $-6 + 3\dot{y} = 0$ or $\dot{y} = 2$. Now the area of the rectangle is $A = xy$ and so we get

$$\frac{dA}{dt} = \dot{x}y + x\dot{y} = 1\cdot 3 + 2\cdot 2 = 7 \text{ square units per second.}$$

5. Find the volume generated by rotating the area between the curves $y = 5x$ and $y = x^2$ for $0 \le x \le 3$ about the $x-$axis. HINT: washers

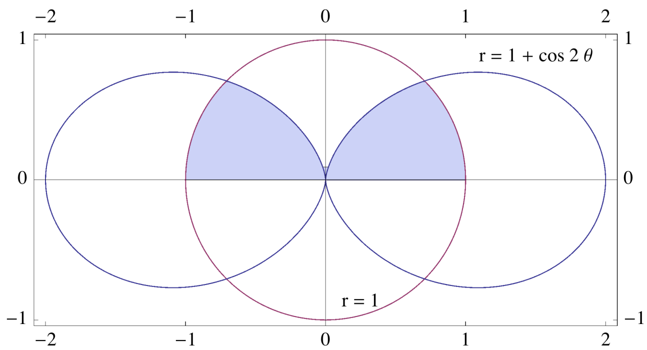


SOLUTION. The volume V is given by integrating the cross sectional area $A(x)$ perpendicular to the x axis over the interval $[0, 3]$. We get

$$V = \int_0^3 A(x)\,dx = \int_0^3 \left((5x)^2 - \left(x^2\right)^2\right)dx = \int_0^3 \left(25x^2 - x^4\right)dx$$

$$= \left(\frac{25\cdot 3^3}{3} - \frac{3^5}{5}\right) = \frac{5^3\cdot 3^3 - 3^6}{15} = \frac{3^3\,(125-27)}{15} = \frac{9\,(98)}{5} = \frac{882}{5}.$$

6. Find the shaded area of the polar region given below



SOLUTION. The area A of the shaded region is

$$A = 2\left(\frac{1}{2}\int_0^{\pi/4} r^2\, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2\, d\theta\right) = \int_0^{\pi/4} 1\, d\theta + \int_{\pi/4}^{\pi/2} (1 + \cos 2\theta)^2\, d\theta$$

since the curves interesect at $\theta = \pi/4$ because

$$1 = r = 1 + \cos 2\theta \text{ implies } \cos 2\theta = 0 \text{ or } \theta = \frac{\pi}{4}.$$

So we get

$$A = \frac{\pi}{4} + \int_{\pi/4}^{\pi/2}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{\pi}{4} + \frac{\pi}{4} + \sin 2\theta\Big|_{\pi/4}^{\pi/2} + \frac{1}{2}\int_{\pi/4}^{\pi/2} (1 + \cos 4\theta)\, d\theta$$

$$= \frac{\pi}{2} - 1 + \frac{\pi}{8} + \frac{\sin 4\theta}{8}\Big|_{\pi/4}^{\pi/2} = \frac{5}{8}\pi - 1.$$