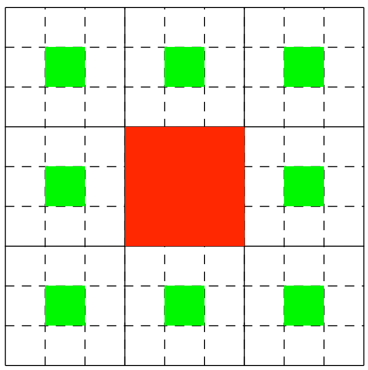
Solutions of the Calculus Contest

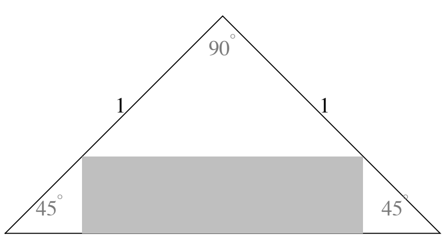
Mathematics Department, University of Cincinnati, May 13, 2008

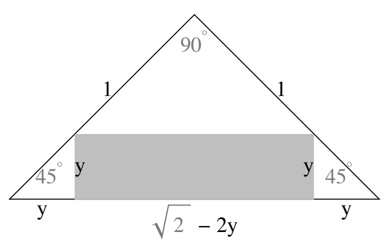
1. The middle third $\left[\frac{1}{3},\ \frac{2}{3}\right] \times \left[\frac{1}{3},\ \frac{2}{3}\right]$ is removed from the unit square $[1,\ 0] \times [1,\ 0]$ creating 8 new squares. The middle third is removed from each of the 8 new squares creating 64 new squares. This process is continued ad infinitum. How much area has been removed?



SOLUTION. The central square S_1 has area $A_1 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. Removal of the center square S_1 creates 8 smaller squares whose central squares S_2 has area $\frac{1}{9} \cdot \frac{1}{9}$ for a total area of $A_2 = \frac{8}{9^2}$. Removal of the squares S_2 creates 8·8 central squares S_3 whose area is $\frac{1}{27}\,\frac{.1}{27}$ for a total area $A_3 = \frac{8^2}{9^3}$. So the area being removed is $\sum A_i = \sum_{i=1}^{\infty} \frac{8^{i-1}}{9^i} = \frac{1}{9} \sum_{i=0}^{\infty} \frac{8^i}{9^i} = \frac{1}{9}\left(\frac{1}{1-8/9}\right) = 1.$

2. Find the dimensions of the rectangle with maximal area in the $45^{\circ} - 45^{\circ} - 90^{\circ}$ right triangle with legs of length 1.





SOLUTION. The area is $A = y\left(\sqrt{2} - 2y\right)$. The maximal area occurs for y with $A' = \sqrt{2} - 4y = 0$ or $y = \sqrt{2}/4$. Thus, the dimensions of the rectangle with maximal area are

$\sqrt{2} - 2\left(\frac{\sqrt{2}}{4}\right) = \frac{\sqrt{2}}{2}$ by $\frac{\sqrt{2}}{4}$ and the maximal area is $A = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{4}\right) = \frac{2}{8} = \frac{1}{4}$.

3. Evaluate the integral $\int_{-\infty}^{\infty} \frac{1}{x^2 - 6x + 10}\, dx$.

SOLUTION. The polynomial $x^2 - 6x + 10$ is irreducible (i.e., does not have real roots). We need to complete the square of the denominator to integrate. We get

$$\int_{-\infty}^{\infty} \frac{1}{x^2 - 6x + 10}\, dx = \int_{-\infty}^{\infty} \frac{1}{x^2 - 6x + + 9 + 1}\, dx = \int_{-\infty}^{\infty} \frac{1}{(x-3)^2 + 1}\, dx = \int_{-\infty}^{\infty} \frac{1}{u^2 + 1}\, du$$

$$= \lim_{\substack{b \to \infty \\ a \to -\infty}} \arctan u \Big|_a^b = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

4. Evaluate $\int_0^1 \sqrt{x - x^2}\, dx$.

SOLUTION. We have

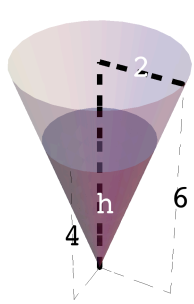
$$\int_0^1 \sqrt{x - x^2}\, dx = \int_0^1 \sqrt{\frac{1}{4} - \frac{1}{4} + x - x^2}\, dx = \int_0^1 \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}\, dx.$$

Set $x - \frac{1}{2} = \frac{1}{2}\sin\theta$. At $x = 1$, $\frac{1}{2}\sin\theta = \frac{1}{2}$ and $\theta = \frac{\pi}{2}$; at $x = 0$, $\frac{1}{2}\sin\theta = -\frac{1}{2}$ and $\theta = -\frac{\pi}{2}$. Also $dx = \frac{1}{2}\cos\theta\, d\theta$. Since $\cos\theta$ is positive on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we get

$$\int_0^1 \sqrt{x - x^2}\, dx = \int_{-\pi/2}^{\pi/2} \sqrt{\frac{1}{4} - \frac{1}{4}\sin^2\theta}\; \frac{1}{2}\cos\theta\, d\theta = \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{1 - \cos 2\theta}{2}\, d\theta = \frac{\pi}{8}.$$

5. Water is poured into a conical cup at the rate of $\frac{2}{3}$ cubic inches per second. If the cup is 6 inches tall and if the top of the cup has a radius of 2 inches, how fast is the water level rising when the water is 4 inches deep?



SOLUTION. This is a related rate problem. We have that

$$\frac{h}{r} = \frac{6}{2} \text{ or } r = \frac{h}{3}.$$

This means that

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h = \frac{\pi h^3}{3^3}.$$

Taking the derivative of both sides with respect to t when $h(t) = 4$ gives

$$\frac{2}{3}\frac{in^3}{s} = V' = \frac{3\pi h^2 h'}{3^3} = \frac{\pi (4)^2 h'}{3^2} in^2$$

so that

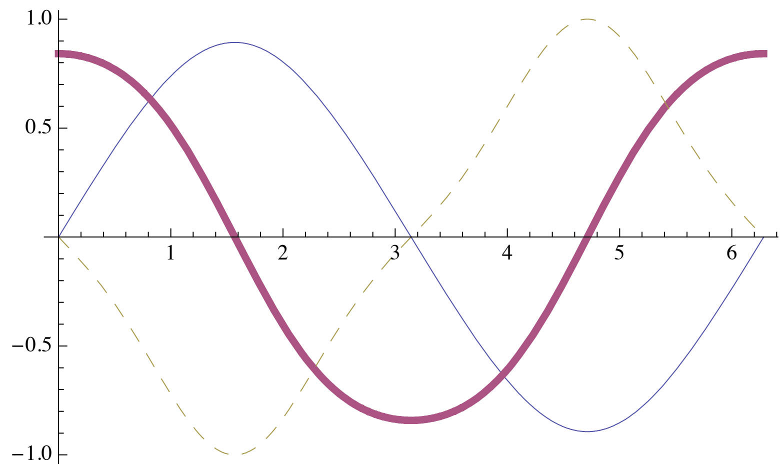
$$h' = \frac{2}{3}\left(\frac{3^2}{4^2 \pi}\right)\frac{in}{sec} = \frac{3}{8\pi}\frac{in}{sec}.$$

6. Find $\int arcsin(3x)\, dx$.

SOLUTION. This requires integration by parts. Let $u = arcsin\, 3x$ and $dv = dx$; then $du = \frac{1}{\sqrt{1-(3x)^2}} \frac{d}{dx}(3x)\, dx = \frac{3}{\sqrt{1-9x^2}} dx$ and $v = x$. We get

$$\int arcsin(3x)\, dx = x\, arcsin\, 3x - \int \frac{3x}{\sqrt{1-9x^2}} dx = x\, arcsin\, 3x + \frac{\sqrt{1-9x^2}}{3}.$$

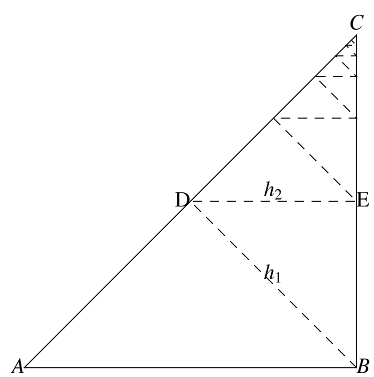
7. Consider the graph of $f,\ f',\ f''$ on the axes below.



Identify the graphs by the labels **Thick, Thin, Dashed**. Write the graphs in order $f,\ f',\ f''$. Why did you choose this order?

SOLUTION. The Thin graph has a maximum at x≈ 1.5 while the Thick graph has a zero at 1.5 and the Dashed graph is negative. So the graphs in order $f,\ f',\ f''$ are **Thin, Thick, Dashed**. Other orders are not possible: 1) Thin, Dashed, Thick does not work at $x = 1.5$ since Thin has a max but Dashed is not 0; 2) Dashed, Thick, Thin does not work on [1.5, 3.1] where Dashed is increasing but Thick is negative; 3) Dashed, Thin, Thick does not work on $[0, 1.5]$ where Dashed is decreasing but Thin is positive; 4) Thick, Dashed, Thin does not work on $[0, 1.5]$ where Thick is concave down but Thin is positive; and 5) Thick, Thin, Dashed does not work [1.5, 3.1] where Thick is concave up but Dashed is negative.

8. The right triangle ΔABC has legs $\overline{AB}$ and $\overline{BC}$ of length 1 and right angle ∠ABC. Draw the altitude h_1 from B to the hypotenuse $\overline{AC}$ to form the new right triangle ΔBDC with right angle ∠BDC. Draw the altitude h_2 from D to the hypotenuse $\overline{BC}$ to form a new right triangle ΔDEC with right angle ∠DEC. This is continued ad infinitum. Find $\sum_{i=1}^{\infty} Length(h_i)$.



SOLUTION. Each h_i is the altitude on a isosceles right triangle with leg h_{i-1} with $h_0 = 1$. In fact, the altitudes bisect the right angles at the vertex producing a 45° angle at the base of the new right triangle. This means that

$$Length(h_i) = Length(h_{i-1}) \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} Length(h_{i-1}).$$

The sum of the lengths is a geometric series, viz.,

$$\sum_{i=1}^{\infty} Length(h_i) = Length(h_1) + \frac{1}{\sqrt{2}} Length(h_1) + \left(\frac{1}{\sqrt{2}}\right)^2 Length(h_1) + \ldots$$

$$= Length(h_1)\left(1 + \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)^2 + \ldots\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{1-\left(1/\sqrt{2}\right)}\right) = \frac{1}{\sqrt{2}-1}.$$