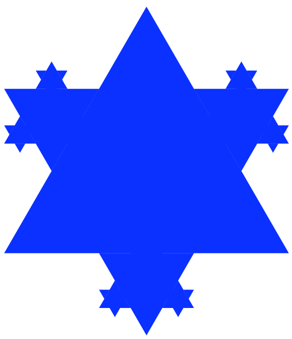
Universty of Cincinnati
Solutions
Calculus Contest 2009

1. Find the area of the snowflake. It starts with an equilateral triangle of side length s. Each side is divided into 3 equal parts and an equilateral triangle with base equal to the middle third of each side is constructed pointing out on the middle third. Then the 2 outward sides of each of the 3 new triangles is divided into 3 equal parts and an equilateral triangle is placed on the two outer sides in the same way. This continued infinitely.



SOLUTION. The area of an equilateral triangle of side s is $A = \frac{1}{2} s^2 \sin \frac{\pi}{3} = \frac{\sqrt{3}}{4} s^2$. In our snowflake, we have one equilateral triangle of side s, 3 equilateral triangles of side $\frac{s}{3}$, 2·3 equilateral triangles of side $\frac{1}{3}\left(\frac{s}{3}\right) = \frac{s}{3^2}$, 2·2·3 equilateral triangles of side $\frac{s}{3^3}$, etc. So we need the area

$$A = \frac{\sqrt{3}}{4}\left(s^2 + 3\left(\frac{s}{3}\right)^2 + 2\cdot 3\left(\frac{s}{3^2}\right)^2 + 2^2\cdot 3\left(\frac{s}{3^3}\right)^2 + \dots\right)$$

$$= \frac{\sqrt{3}}{4} s^2\left(1 + \frac{1}{3} + \frac{2}{3^3} + \frac{2^2}{3^5} + \dots\right) =$$

$$= \frac{\sqrt{3}}{4} s^2\left(1 + \frac{1}{3}\left(1 + \frac{2}{9} + \left(\frac{2}{9}\right)^2 + \dots\right) = \frac{\sqrt{3}}{4} s^2\left(1 + \frac{1}{3}\left(\frac{1}{1-2/9}\right)\right) = \frac{5\sqrt{3}}{14} s^2.$$

2. Calculate $\int_0^{\pi/2} \sqrt{1 + \sin x}\, dx$. HINT: $\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1$.

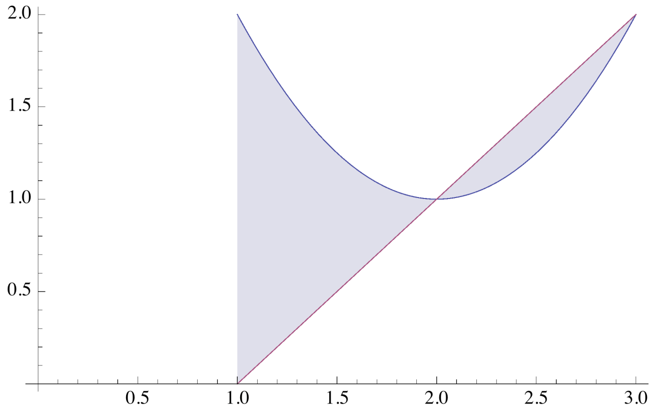
SOLUTION. We have

$$\int_0^{\pi/2} \sqrt{1 + \sin x}\, dx = \int_0^{\pi/2} \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}\, dx \qquad \left(\text{since } \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}\right)$$

$$= \int_0^{\pi/2} \left(\sin \frac{x}{2} + \cos \frac{x}{2}\right) dx \qquad \left(\text{since } \sin \frac{x}{2} + \cos \frac{x}{2} \geq 0 \text{ for } 0 \leq x \leq \pi/2\right)$$

$$= 2\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)\Big|_0^{\pi/2} = 2.$$

3. Let $f(x) = (x-2)^2 + 1$ and $g(x) = x - 1$ on $[1, 3]$. Find the volume obtained by



a. Rotating the area between the curves around the y-axis.

SOLUTION. We need to find where the curves intersect. We have

$$(x-2)^2 + 1 - (x-1) = x^2 - 5x + 6 = (x-3)(x-2) = 0$$

SOLUTION ROTATE ABOUT Y AXIS.

$$\int_1^2 2\pi x\left(x^2 - 5x + 6\right) dx - \int_2^3 2\pi x\left(x^2 - 5x + 6\right) dx$$

$$= 2\pi\left(\int_1^2 \left(x^3 - 5x^2 + 6x\right) dx - \int_2^3 \left(x^3 - 5x^2 + 6x\right) dx\right)$$

$$= 2\pi\left(\left(\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right)\Big|_1^2 - \left(\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right)\Big|_2^3\right)$$

$$= 2\pi\left(2\left(\frac{2^4}{4} - \frac{5}{3}2^3 + 3\cdot 2^2\right) - \left(\left(\frac{1^4}{4} - \frac{5}{3}1^3 + 3\cdot 1^2\right) + \left(\frac{3^4}{4} - \frac{5}{3}3^3 + 3\cdot 3^2\right)\right)\right)$$

$$= 3\pi$$

b. Find the volume obtained by rotating the area between the curves about the line $y = -1$.

SOLUTION ROTATE ABOUT Line y = -1.

Volume between $[1, 2]$

$$V_1 = \int_1^2 \pi\left((x-2)^\wedge 2 + 1 + 1\right)^2 dx - \int_1^2 \pi\,(x - 1 + 1)^2\, dx$$

$$= \pi \int_1^2 \left(\left(x^2 - 4x + 6\right)^2 - x^2\right) dx$$

$$= \pi \int_1^2 \left(x^4 - 8x^3 + 27x^2 - 48x + 36\right) dx$$

$$= \pi\left(\frac{x^5}{5} - 2x^4 + 9x^3 - 24x^2 + 36x\right)\Big|_1^2$$

$$= \pi\left(\left(\frac{32}{5} - 32 + 72 - 96 + 72\right) - \left(\frac{1^5}{5} - 2 + 9 - 24 + 36\right)\right)$$

$$= \pi\left(\frac{31}{5} - 3\right) = \frac{16}{5}\pi.$$

4. Find a polynomial $p(x)$ such that $x\,p''(x) + p(x) \equiv x^2 + 1$.

SOLUTION. The highest degree for a test polynomial $p(x)$ is 2. Any other polynomial with highest power x^n would give a left side of the equation a nonzero term involving x^n. So we let $p(x) = a x^2 + b x + c$ be the test polynomial. Then we get

$$2ax + ax^2 + bx + c = ax^2 + (2a + b)x + c = x^2 + 1.$$

Identitying coefficients with like degree on both sides give the linear equations $a = 1,\ 2a + b = 0,\ c = 1$. This means that $b = -2$ and the polynomial $p(x) = x^2 - 2x + 1$.

5. Let $a > 0$.

a) Show that $x^a > 2 \ln x$ for all x sufficiently large. HINT: $\lim_{x \to \infty} \frac{x^a}{\ln x} = \infty$.

SOLUTION. We have that

$$\lim_{x \to \infty} \frac{x^a}{\ln x} \overset{\infty/\infty}{=} \lim_{x \to \infty} \frac{a\,x^{a-1}}{\frac{1}{x}} = \lim_{x \to \infty} a\,x^a = \infty.$$

by l'Hôpital's rule. This means that there is an N such that $a > N$ implies $\frac{x^a}{\ln x} > 2$ or $a > N$ implies $x^a > 2 \ln x$.

b) Show that the integral $\int_1^\infty e^{-x^2}\,dx$ converges.

SOLUTION. For $x \geq 1, x^2 \geq x$ and $-x^2 \leq -x$. So we have that $e^{-x^2} \leq e^{-x}$ for $x \geq 1$. This means that

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b \to \infty} \int_1^b e^{-x^2}\,dx \leq \lim_{b \to \infty} \int_1^b e^{-x}\,dx = \frac{1}{e} < \infty.$$

To use the hint in part a), we have that $2 \ln x < x^2$ for x sufficiently large, say $x > N$. So we get that $-x^2 < -2 \ln x$ for x large and $e^{-x^2} < e^{-2 \ln x} = x^{-2}$ for x large. This means that

$$\int_N^\infty e^{-x^2}\,dx = \lim_{b \to \infty} \int_N^b e^{-x^2}\,dx \leq \lim_{b \to \infty} \int_N^b x^{-2}\,dx = \frac{1}{N} < \infty$$

and x converges.

6. If $y = x^3 + x$ and $u = y^2 - 1$, find $\frac{dx}{du}$.

SOLUTION. Using the chain rule, we get

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = 2y\left(3x^2 + 1\right) = 2\left(x^3 + x\right)\left(3x^2 + 1\right)$$

and

$$\frac{dx}{du} = \frac{1}{du/dx} = \frac{1}{2\left(x^3 + x\right)\left(3x^2 + 1\right)}.$$

7. Evaluate $\int_2^\infty \frac{dt}{t(\ln t)^3}$.

SOLUTION. Using the substitution $u = \ln t$, we get

$$\int_2^\infty \frac{dt}{t(\ln t)^3} = \int_{\ln 2}^\infty \frac{du}{u^3} = \lim_{b \to \infty} -\frac{1}{2}\,u^{-2}\Big|_{\ln 2}^{b} = \frac{1}{2(\ln 2)^2}.$$