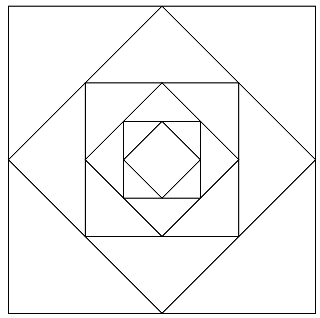
Calculus Contest 2010

1. Given a square S and new square S' is formed by connecting the consecutive midpoints of S. Starting with the unit square S with vertices (0, 0), (1, 0), (1, 1), (0, 1), new squares S', $(S')' = S''$, S''', … are formed. What is the sum of the perimeters of all the squares?



SOLUTION. If s is the length of the side of the square, then the inscribed square has side of length $\frac{s}{2}\sqrt{2}$. So we need to find

$$4 + 4\left(\frac{\sqrt{2}}{2}\right) + 4\left(\frac{\sqrt{2}}{2}\right)^2 + \ldots = \frac{4}{1-\frac{\sqrt{2}}{2}} = \frac{8}{2-\sqrt{2}}.$$

2. Approximate the integral $\int_0^{1/2} e^{-t^2}\,dt$ to 4 decimal places.

SOLUTION. The exponential has the Taylor expansion

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\ldots$$

and

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} + \ldots$$

with

$$\int_0^{1/2} e^{-t^2}\,dt = \left(t - \frac{t^3}{3} + \frac{t^5}{5\cdot 2!} - \frac{t^7}{7\cdot 3!} + \frac{t^9}{9\cdot 4!}\ldots\right)\Big|_0^{1/2} = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5\cdot 2!}\left(\frac{1}{2}\right)^5 - \frac{1}{7\cdot 3!}\left(\frac{1}{2}\right)^7 + \frac{1}{9\cdot 4!}\left(\frac{1}{2}\right)^9 + \ldots.$$

Since the solution series is an alternating series, the first omitted term can serve as the error term. We have that

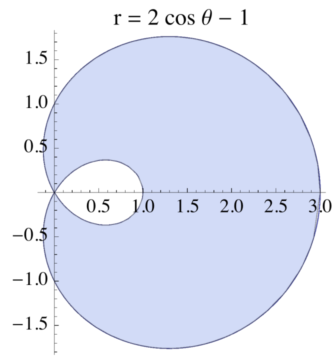
$$\frac{1}{5\cdot 2!}\left(\frac{1}{2}\right)^5 = \frac{1}{10}\cdot\frac{1}{32} = \frac{1}{320} > .0001,$$

$$\frac{1}{7\cdot 3!}\left(\frac{1}{2}\right)^7 = \frac{1}{42}\cdot\frac{1}{128} > \frac{1}{50}\cdot\frac{1}{200} = \frac{1}{10000} = .0001$$

$$\frac{1}{9\cdot 4!}\left(\frac{1}{2}\right)^9 = \frac{1}{9}\cdot\frac{1}{24}\cdot\frac{1}{512} < \frac{1}{10000} = .0001.$$

So $\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5\cdot 2!}\left(\frac{1}{2}\right)^5 - \frac{1}{7\cdot 3!}\left(\frac{1}{2}\right)^7 = \frac{4133}{8960} \approx 0.4613$ or .4612 is a good approximation.

3. Find the shaded area in the polar diagram below



SOLUTION. The area of the inner loop is

$$A_{\text{in}} = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(2\cos\theta - 1)^2\, d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left(4\left(\frac{1+\cos 2\theta}{2}\right) - 4\cos\theta + 1\right) d\theta$$

$$= \frac{1}{2}(3\theta + \sin 2\theta - 4\sin\theta)\,\Big|_{-\pi/3}^{\pi/3} = \frac{1}{2}\left(2\pi + 2\left(\frac{\sqrt{3}}{2}\right) - 4\sqrt{3}\right) = \frac{1}{2}\left(2\pi - 3\sqrt{3}\right)$$

while the area of the outer loop is

$$A_{\text{out}} = \frac{1}{2}\int_{-2\pi/3}^{2\pi/3}(2\cos\theta - 1)^2\, d\theta$$

$$= \frac{1}{2}(3\theta + \sin 2\theta - 4\sin\theta)\,\Big|_{-2\pi/3}^{2\pi/3}$$

$$= \frac{1}{2}\left(4\pi - \sqrt{3} - 4\sqrt{3}\right)$$

and the area is

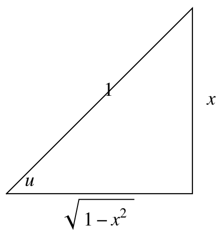
$$A_{\text{out}} - A_{\text{in}} = \frac{1}{2}\left(4\pi - 5\sqrt{3}\right) - \frac{1}{2}\left(2\pi - 3\sqrt{3}\right) = \pi - \sqrt{3}\,.$$

4. Find $\displaystyle\int \frac{dx}{x^2\sqrt{1-x^2}}$.

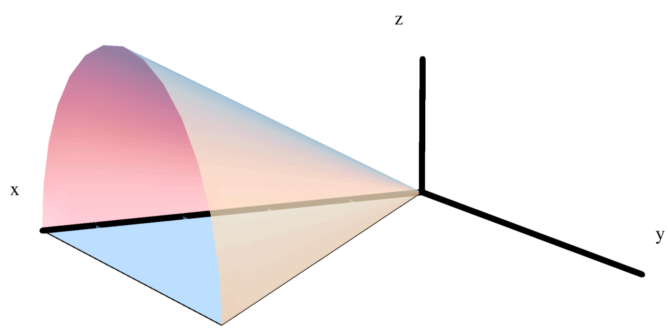
SOLUTION. Seting $x = \sin u$, we get

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \int \frac{d\sin u}{\sin^2 u\sqrt{1-\sin^2 u}} = \int \frac{\cos u\, du}{\cos u \sin^2 u} = \int \csc^2 u\, du = -\cot u + C = -\frac{\sqrt{1-x^2}}{x} + C$$

Using the triangle below we get that $-\cot u = -\frac{\text{adjacent}}{\text{opposite}} = -\frac{\sqrt{1-x^2}}{x}$



5. Find the volume of the solid whose base is the triangle bounded by the lines $y = x$, $y = 0$, $x = 3$ and whose cross sections perpendicular to the x-axis are semicircles.



SOLUTION. The volume is

$$V = \int_0^3 A(x)\, dx$$

where $A(x)$ is the cross sectional area perpendicular to the x-axis at x. Here $A(x)$ is half a disk with diameter x. So $A(x) = \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \frac{\pi x^2}{8}$. So the volume is

$$V = \frac{\pi x^3}{24}\Big|_0^3 = \frac{9}{8}\pi.$$

6. Find the equation of the tangent line to the lemniscate $2\left(x^2 + y^2\right)^2 = 25\left(x^2 - y^2\right)$ at (3, 1).

SOLUTION. We get the derivative by implicit differentiation:

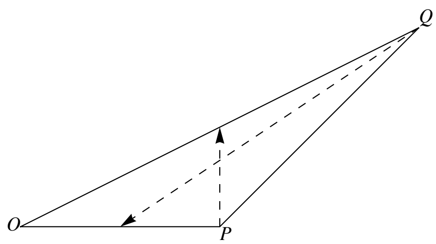
$$4\left(x^2 + y^2\right)(2x + 2yy') = 25(2x - 2yy').$$

Substituting $x = 3$ and $y = 1$, we get

$4(9 + 1)(6 + 2y') = 25(6 - 2y')$ or $80y' + 50y' = -240 + 150$ or $y' = -\frac{9}{13}$. The equation of the tangent line is

$$\frac{-9}{13}(x - 3) = y - 1 \text{ or } 9x + 13y = 40$$

7. Use vectors methods to find the intersection of the two dashed medians of the triangle below. Assume O is the orgin



SOLUTION. The median from P has the direction vector $d_1 = \frac{1}{2} Q - P$ and the median from Q has the direction vector $d_2 = \frac{1}{2} P - Q$. The median lines have parametric equations $s\, d_1 + P$ and $t\, d_2 + Q$. So we need s and t with

$$s\left(\frac{1}{2} Q - P\right) + P = s\ d_1 + P = t\, d_2 + Q = t\left(\frac{1}{2} P - Q\right) + Q$$

or

$$\left(\frac{1}{2} s + t\right) Q - \left(\frac{1}{2} t + s\right) P = Q - P.$$

Since P and Q are not parallel, we get $\frac{1}{2} s + t = 1$ and $\frac{1}{2} t + s = 1$. The simultaneous equations

$$\begin{aligned} s + 2t &= 2 \\ 2s + t &= 2 \end{aligned}$$

have the solutions $3t = 2$ or $t = 2/3$ and similarly $s = 2/3$. Either of these is sufficient to get the point of intersetion

$$\text{Intersection} = s\left(\frac{1}{2} Q - P\right) + P\,\Big|_{s=2/3} = \frac{1}{3} Q + \frac{1}{3} P$$

or

$$\text{Intersection} = t\left(\frac{1}{2} P - Q\right) + Q\,\Big|_{t=2/3} = \frac{1}{3} Q + \frac{1}{3} P$$