Department of Mathemtical Sciences
Competitive Calculus Examination

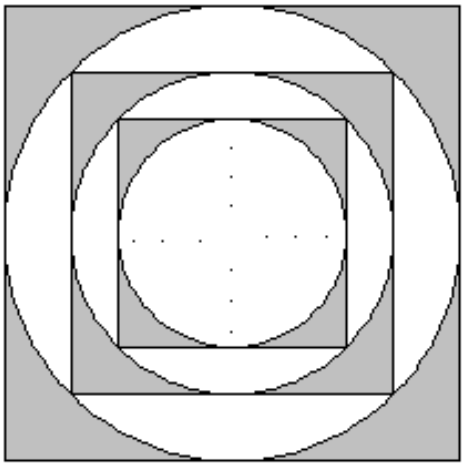
May 20, 1997

Directions: Show your work and express yourself in a legible and logical fashion. Full credit will only be awarded to answers supported by clear explanations. weight.

1. Find the volume of the largest (right circular) cone that can be inscribed in a sphere of radius a.

2. Consider the quartic curves of the form $y = ax^4 + bx^3 + cx^2$ where $a > 0$. For what values of a, b, and c will this curve have 0, 1, 2, or 3 critical points? For each of these situations, what combinations of relative maxima, relative minima and inflection points are possible. (For instance if there are 3 critical points is it possible to have two inflection points and one relative maximum?)

3. Find the shaded area below.



4. (a) Show that the series $\sum n^2/2^n$ converges.

   (b) Find the exact value of $\sum_{n=0}^{\infty} n^2/2^n$. (Hint: Start with the power series expression for $1/(1-x)$. By successively differentiating and multiplying by x find an expression for $\sum_{n=0}^{\infty} n^2 x^n$.)

5. (a) Show that

$$\frac{1}{n}\ln\left(\frac{k-1}{n}\right) < \int_{(k-1)/n}^{k/n} \ln x\, dx < \frac{1}{n}\ln\left(\frac{k}{n}\right)$$

   for $k = 2,\ 3, \ldots, n$.

   (b) Deduce from the previous assertion that

$$\ln\left[\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\cdots\left(\frac{n-1}{n}\right)\right]^{1/n} < \int_{1/n}^{1} \ln x\, dx < \ln\left[\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\cdots\left(\frac{n}{n}\right)\right]^{1/n}.$$

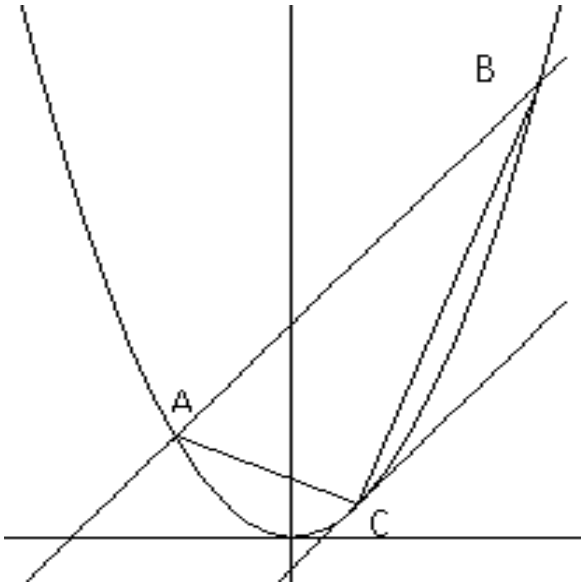
   (c) Deduce that

$$-1 + \frac{1}{n} < \ln\left[\frac{n!}{n^n}\right]^{1/n} < -1 + \frac{1}{n} + \frac{\ln n}{n}.$$

(d) Deduce that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

6. Suppose that a line intersects the parabola $y = x^2$ in two points A and B as shown in the figure. Let C be the point on the parabola where the tangent line is parallel to the line through A and B. Show that the area of the parabolic segment cut off by the line is four-thirds the area of triangle ABC.



7. Two cylinders are inscribed in a cube of side length 2 as illustrated below. What is the volume of the region that the two cylinders enclose? (Hint: Use horizontal slices.)