Department of Mathematical Sciences
Competitive Calculus Examination

May 20, 1998

Directions: *Show your work and express yourself in a legible and logical fashion. Full credit will only be awarded to answers supported by clear explanations. All questions carry equal weight. The later questions tend to be more challenging than the earlier ones. You may answer as many questions as you like. The use of graphing calculators is not permitted.*

1. Show that
$$\sum_{n=10}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)^2}$$
converges but that
$$\sum_{n=10}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$$
does not.

2. Show that if $t > 0$,
$$\int_0^{\infty} e^{-tx} \sin x \, dx = \frac{1}{1+t^2}$$

3. A beetle travels 5 meters due East. It turns 90° to the left and travels a fraction p of her original distance. Travel continues in this way forever: turn left, travel the fraction p of the previous distance. To what point does the beetle's path spiral?

4. Prove that the function
$$\frac{1-\cos x}{\sin(x-a)} \quad 0 < a < \pi$$
has infinitely many relative maxima equal to 0 and infinitely many relative minima equal to $2 \sin a$. (Hint: $\cos(u - v) = \sin u \sin v + \cos u \cos v$)

5. A dog waiting at a corner of a square swimming pool can run at speed u or swim at speed v. Find the shortest time in which he can reach his owner at the opposite corner of the pool.

6. Let A and B be points on the parabola $y = x^2$. Find the position of the point P lying on the parabola between A and B for which the area of the triangle APB is as large as possible.

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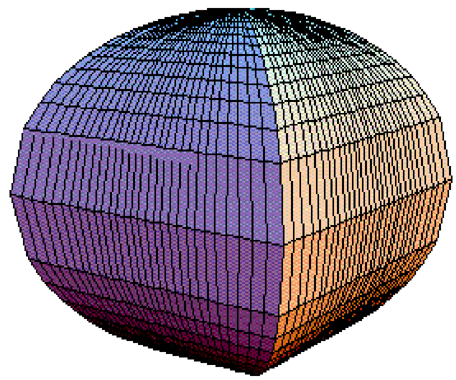
7. Find the volume of the region common to two circular cylinders of radius r whose axes meet at right angles.



8. Prove that
$$\int_0^1 \frac{t^4(1-t)^4}{1+t^2}\,dt = \frac{22}{7} - \pi.$$
Evaluate $\int_0^1 t^4(1-t)^4\,dt$ and deduce that
$$\frac{22}{7} - \frac{1}{1260} > \pi > \frac{22}{7} - \frac{1}{630}$$