Name:________________________ M#:__________ Instructor:____________________

Instructions: This exam has seven problems on seven pages. Show all your work, expressing yourself in clear and concise manner. Do as many problems as you can, but be advised that a complete solution to a problem may be worth more than several partial ones. Use the backs of the exam pages for work, if necessary. No calculators of any kind are allowed.

1. Suppose that f'' is continuous on $[0, \pi]$ and that

$$\int_0^\pi [f(x) + f''(x)] \sin x \, dx = 2.$$

Given that $f(\pi) = 3$, compute $f(0)$.

Solution: Since f'' is continuous, so are f and f', and we can integrate by parts:

$$\int_0^\pi f(x) \sin x \, dx = -f(x) \cos x \Big|_0^\pi + \int_0^\pi f'(x) \cos x \, dx = f(\pi) + f(0) + \int_0^\pi f'(x) \cos x \, dx$$

and

$$\int_0^\pi f''(x) \sin x \, dx = f'(x) \sin x \Big|_0^\pi - \int_0^\pi f'(x) \cos x \, dx = -\int_0^\pi f'(x) \cos x \, dx.$$

Putting the two integrals together gives

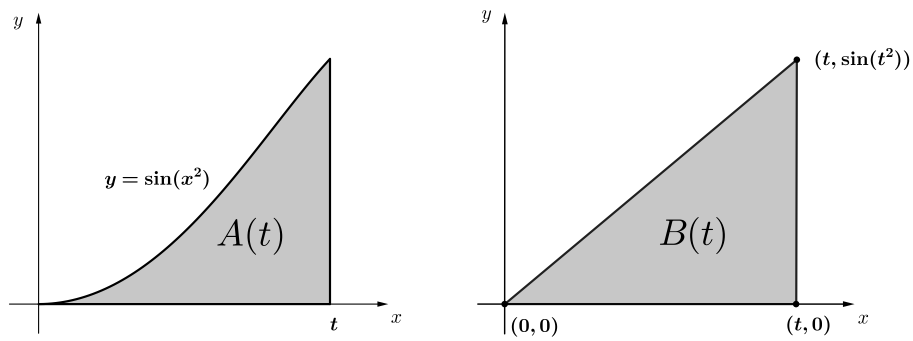
$$\int_0^\pi [f(x) + f''(x)] \sin x \, dx = f(\pi) + f(0) = 2.$$

Since $f(\pi) = 3$, we conclude that $f(0) = -1$.

2. Let $A(t)$ be the area under the curve $y = \sin(x^2)$, $0 \leq x \leq t$. Let $B(t)$ be the area of the triangle with the vertices $(0,0)$, $(t, \sin(t^2))$, and $(t, 0)$. Find $\lim_{t \to 0^+} \dfrac{A(t)}{B(t)}$.

Solution: We have

$$A(t) = \int_0^t \sin(x^2)\,dx, \quad B(t) = \frac{1}{2}\,t\sin(t^2),$$



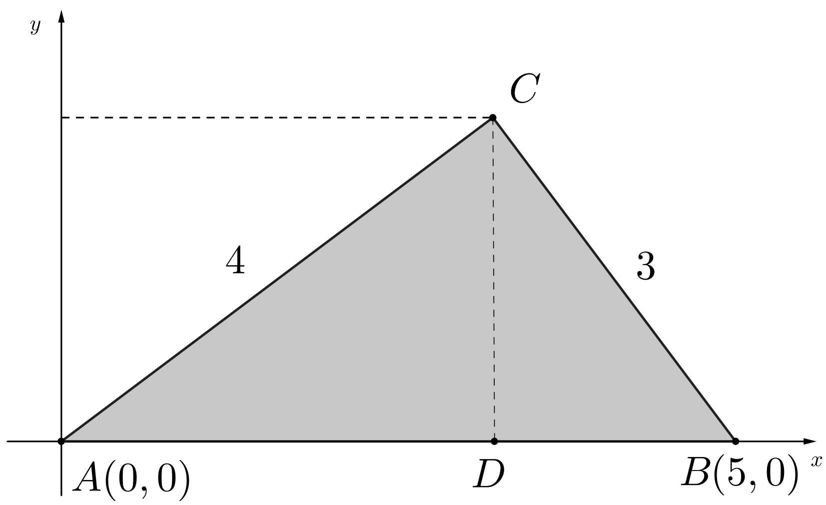
Observe that $\lim_{t\to 0^+} A(t) = 0$ and $\lim_{t\to 0^+} B(t) = 0$. Therefore, by L'Hôspital's Rule and the Fundamental Theorem of Calculus,

$$\lim_{t\to 0^+} \frac{A(t)}{B(t)} = \lim_{t\to 0^+} \frac{A'(t)}{B'(t)} = \lim_{t\to 0^+} \frac{\sin(t^2)}{\frac{1}{2}\sin(t^2) + t^2\cos(t^2)}$$

$$= \lim_{t\to 0^+} \frac{\frac{\sin(t^2)}{t^2}}{\frac{1}{2}\frac{\sin(t^2)}{t^2} + \cos(t^2)} = \frac{1}{\frac{1}{2}+1} = \frac{2}{3}.$$

3. A right triangle whose three sides have lengths 3, 4, and 5 ft is rotated about its hypotenuse. Compute the area of the resulting surface of revolution.

Solution: Position the triangle on a coordinate system so that its hypotenuse AB is along the x-axis, with $A = (0, 0)$ and $B = (5, 0)$ (see the figure). We first need to determine the coordinates of the point C, that is the lengths $|AD|$ and $|CD|$.



We have

$$\frac{|AD|}{|AC|} = \frac{|AC|}{|AB|} = \frac{|CD|}{|CB|} \Longrightarrow |AD| = \frac{|AC|^2}{|AB|} = \frac{16}{5}, \quad |CD| = \frac{|AC||CB|}{|AB|} = \frac{12}{5}.$$

Therefore, the slope of the line through A and C is equal to $\frac{|CD|}{|AD|} = \frac{12/5}{16/5} = \frac{3}{4}$ and the line itself is given by $y = \frac{3}{4}x$. Similarly, the slope of the line through B and C is equal to $-\frac{|CD|}{|DB|} = -\frac{12/5}{9/15} = -\frac{4}{3}$ and the line itself is $y = -\frac{4}{3}(x-5)$. Using the formula for the area of the surface of revolution, we obtain

$$S = 2\pi \int_0^{16/5} \frac{3}{4}x \sqrt{1 + \left(\frac{3}{4}\right)^2}\, dx + 2\pi \int_{16/5}^{5} \left(-\frac{4}{3}(x-5)\right) \sqrt{1 + \left(\frac{4}{3}\right)^2}\, dx$$

$$= 2\pi \left[\frac{15}{16} \int_0^{16/5} x\, dx - \frac{20}{9} \int_{16/5}^{5} (x-5)\, dx \right]$$

$$= 2\pi \left[\frac{15}{16} \cdot \frac{1}{2} \cdot \left(\frac{16}{5}\right)^2 + \frac{20}{9} \cdot \frac{1}{2} \cdot \left(\frac{9}{5}\right)^2 \right] = \frac{84}{5}\,\pi.$$

4. Let $f(x) = \displaystyle\int_1^x \frac{\ln t}{1+t}\,dt$ for $x > 0$. Find a formula for $f(x) + f(\frac{1}{x})$ that does not involve integrals.

Solution:

Solution 1: Using the substitution $z = 1/x$, we have

$$f(1/x) = \int_1^{1/x} \frac{\ln t}{1+t}\,dt = \int_1^x \frac{-\ln z}{1+1/z}\Big(-\frac{1}{z^2}\Big)\,dz = \int_1^x \frac{\ln z}{1+z}\Big(\frac{1}{z}\Big)\,dz,$$

and so

$$f(x) + f(1/x) = \int_1^x \frac{\ln t}{1+t}\,dt + \int_1^x \frac{\ln t}{1+t}\Big(\frac{1}{t}\Big)\,dt$$

$$= \int_1^x \frac{\ln t}{1+t}\Big(1+\frac{1}{t}\Big)\,dt = \int_1^x \frac{\ln t}{t}\,dt = \frac{1}{2}\ln^2 x.$$

Solution 2: This is a reformulation of Solution 1 using the Fundamental Theorem of Calculus:

$$[f(x)+f(1/x)]' = f'(x) - f'(1/x)\,\frac{1}{x^2} = \frac{\ln x}{1+x} - \frac{\ln(1/x)}{1+(1/x)}\,\frac{1}{x^2} = \frac{\ln x}{x},$$

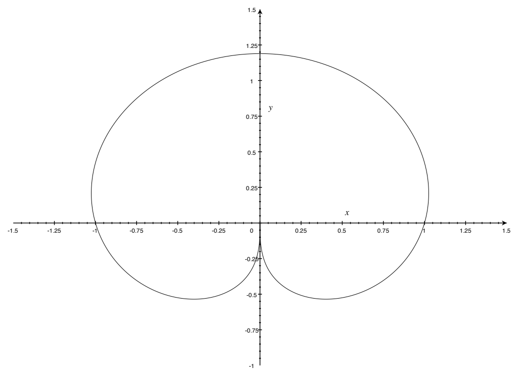
which means that

$$f(x) + f(1/x) = \int \frac{\ln x}{x}\,dx = \frac{1}{2}\ln^2 x + C.$$

Setting $x = 1$ gives $2f(1) = C$. By the definition of f, we have $f(1) = 0$, so $C = 0$ and

$$f(x) + f(1/x) = \frac{1}{2}\ln^2 x.$$

5. Find the area of the region enclosed by the polar curve $r(\theta) = (1+\sin\theta)^{1/4},\ 0 \leq \theta \leq 2\pi$. (Hint for your integral: $1 = \sin^2(\theta/2) + \cos^2(\theta/2)$.)



Solution: We have $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} r^2(\theta)\, d\theta = \dfrac{1}{2}\displaystyle\int_0^{2\pi} (1+\sin\theta)^{1/2} d\theta.$

Now, $1+\sin\theta = \sin^2(\theta/2) + \cos^2(\theta/2) + 2\sin(\theta/2)\cos(\theta/2) = (\sin(\theta/2) + \cos(\theta/2))^2$ and so

$$A = \frac{1}{2}\int_0^{2\pi} |\sin(\theta/2) + \cos(\theta/2)|\, d\theta$$

$$= \int_0^{\pi} |\sin t + \cos t|\, dt = (-\cos t + \sin t)\Big|_0^{3\pi/4} - (-\cos t + \sin t)\Big|_{3\pi/4}^{\pi}$$

$$= 2\sqrt{2}.$$

6. (a) Find the Maclaurin series for $\cos^2(x)$ and state its radius of convergence. (Hint: no need to square a power series.)

Solution: We have

$$\cos^2 x = \frac{1}{2}(1+\cos(2x)) = \frac{1}{2}\left[1+\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}\right] = 1+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}x^{2n}}{(2n)!}.$$

This series inherits its radius of convergence from the series for $\cos x$, thus the radius is ∞.

(b) Use the fact that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \quad \text{and} \quad \int_{-\infty}^{\infty} x^{2n}e^{-x^2}\,dx = \frac{(2n-1)!}{2^{2n-1}(n-1)!}\sqrt{\pi},\ n\geq 1,$$

together with your result for part (a), to show that

$$\int_{-\infty}^{\infty} e^{-x^2}\cos^2(x)\,dx = \frac{(1+e)\sqrt{\pi}}{2e}.$$

Solution: We can multiply a convergent series by a number, term-by-term:

$$e^{-x^2}\cos^2 x = e^{-x^2}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}}{(2n)!}\,x^{2n}e^{-x^2},$$

and then integrate it term-by-term (within the interval of convergence, which is $(-\infty,\infty)$ in our case):

$$\int_{-\infty}^{\infty} e^{-x^2}\cos^2 x\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}}{(2n)!}\int_{-\infty}^{\infty} x^{2n}\,e^{-x^2}\,dx$$

$$= \sqrt{\pi}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}}{(2n)!}\,\frac{(2n-1)!}{2^{2n-1}(n-1)!}\sqrt{\pi}$$

$$= \sqrt{\pi}+\frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}.$$

Recall that $e^{-1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$, so $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!} = e^{-1}-1$. This gives

$$\int_{-\infty}^{\infty} e^{-x^2}\cos^2 x\,dx = \sqrt{\pi}+\frac{\sqrt{\pi}}{2}\left(\frac{1}{e}-1\right) = \frac{(1+e)\sqrt{\pi}}{2e}.$$

□

7. Let f be a continuous function on the interval $[0, 1]$ such that $\int_0^1 f(t)\,dt = 0$. Show that

$$\int_0^1 e^{af(t)}dt \geq 1$$

for any real number a.

(Hint: treat the left-hand side of this inequality as a function of a and examine its derivative(s), in particular at zero. Assume that you can differentiate under the integral sign.)

Solution: Let $g(a) = \int_0^1 e^{af(t)}dt$. Then

$$g'(a) = \frac{d}{da}\int_0^1 e^{af(t)}dt = \int_0^1 \frac{d}{da}\left(e^{af(t)}\right)dt = \int_0^1 f(t)e^{af(t)}dt.$$

Similarly,

$$g''(a) = \int_0^1 [f(t)]^2 e^{af(t)}dt.$$

We have

$$g(0) = \int_0^1 dt = 1, \quad g'(0) = \int_0^1 f(t)\,dt = 0.$$

Furthermore, since f is continuous on $[0, 1]$, we have

$$g''(a) = \int_0^1 [f(t)]^2 e^{af(t)}dt > 0 \quad \text{for any } a,$$

unless f is identically zero on $[0, 1]$. (If f is identically zero on $[0, 1]$, the statement to be shown follows trivially since in that case $g(a) = 1$ for all a.)

Therefore, by the second derivative test the point $a = 1$ is a point of global minimum of g. Since $g(0) = 1$, we have $g(a) \geq 1$ for all a, i.e.

$$\int_0^1 e^{af(t)}dt \geq 1,$$

as required. □

(Alternatively, one can use Taylor's formula:

$$g(a) = g(0) + g'(0)a + \frac{1}{2}\,g''(c)a^2 = 1 + \frac{1}{2}\,g''(c)a^2,$$

for some c between 0 and a. Since $g'' \geq 0$, we have $g(a) \geq 1$.)