UC Calculus Contest April 3, 2014

Name:_____ **M#:**_____ **Instructor:**_____

Instructions: *This exam has seven problems on seven pages. Show all your work, expressing yourself in clear and concise manner. Do as many problems as you can, but be advised that a complete solution to a problem may be worth more than several partial ones. Use the backs of the exam pages for work, if necessary. No calculators of any kind are allowed.*

1

A right circular cone is inscribed in a sphere of radius R as in Figure 1. Find the maximal possible cone volume. What is the ratio between the sphere volume and the maximal inscribed cone volume?

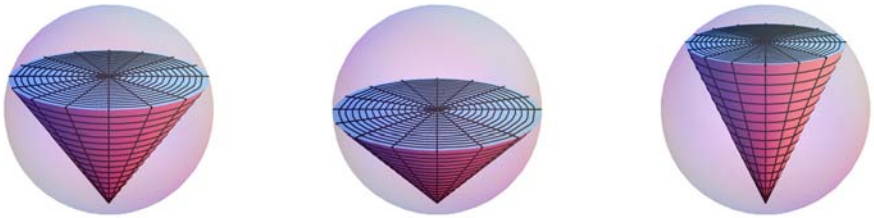

Figure 1.

2

One corner of a page of width $a = 8$ inches is folded over to just reach the opposite side as indicated in Figure 2. After expressing the length L of the crease in terms the angle θ, find the width x of the part folded over when L is a minimum.

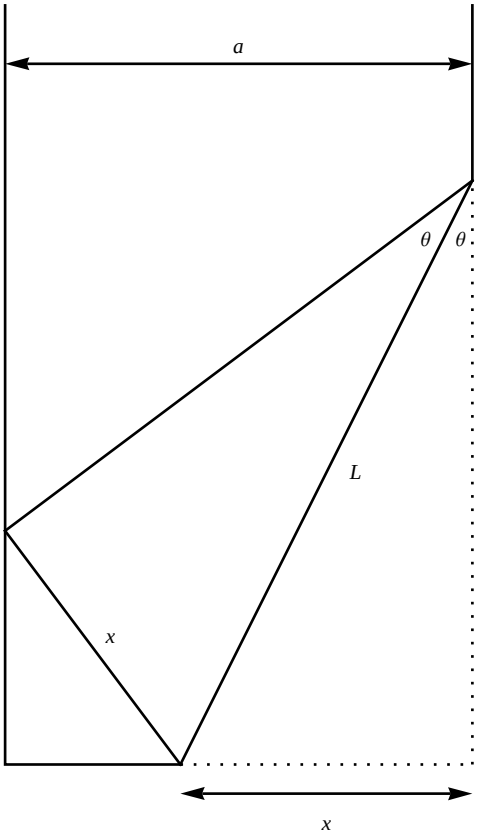

Figure 2.

3

Suppose that a_1, a_2, ..., a_n are real numbers such that the function

$$f(x) = a_1 \sin(x) + a_2 \sin(2x) + ... + a_n \sin(nx) \tag{3.1}$$

satisfies $|f(x)| \leq |\sin(x)|$, for all real numbers x. Prove that

$$|a_1 + 2a_2 + ... + na_n| \leq 1 \tag{3.2}$$

4

For what values of p does the series $\sum_{n=6}^{\infty}\left(e^{-\frac{1}{n^2}}+\frac{1}{n^2}-1\right)^p$ converge? Fully justify your answer.

5

Show that the limit $\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right)$ exists, and find an upper and lower bound for γ.

6

Calculate the following integral and check your result by differentiation.

$$\int (\ln x)^2 \, dx \tag{6.1}$$

7

Calculate the following integral and check your result by differentiation.

$$\int \frac{1}{x^7 - x}\, dx \tag{7.1}$$