UC Calculus Contest April 3, 2014

Name:____________________ **M#:**______________ **Instructor:**_________________

Instructions: *This exam has seven problems on seven pages. Show all your work, expressing yourself in clear and concise manner. Do as many problems as you can, but be advised that a complete solution to a problem may be worth more than several partial ones. Use the backs of the exam pages for work, if necessary. No calculators of any kind are allowed.*

1

A right circular cone is inscribed in a sphere of radius R as in Figure 1. Find the maximal possible cone volume. What is the ratio between the sphere volume and the maximal inscribed cone volume?

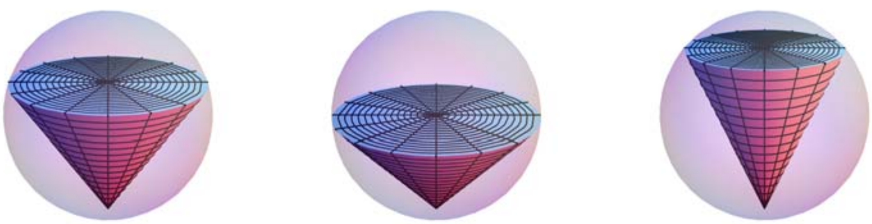

Figure 1.

1.1 solution

Let αR, for $0 \le \alpha \le 1$, be the distance the origin and the cone-base (so that the hight of the cone is equal to $R + \alpha R$). Setting $z = \alpha R$, $x = r\cos(\theta)$, $y = r\sin(\theta)$ in $x^2 + y^2 + z^2 = R^2$, we get

$$r = R\sqrt{1-\alpha^2}$$

so that the volume of the cone, as a function of α, is equal to

$$V(\alpha) = \frac{(R + \alpha R)\, r^2 \pi}{3} = \frac{1}{3}\pi R^3 (1+\alpha)\left(1-\alpha^2\right)$$

Solving equation $V'(\alpha) = 0$, we get

$$\alpha_1 = -1,\ \alpha_2 = \frac{1}{3}$$

Since $0 \le \alpha \le 1$, we get $\alpha = \alpha_2 = 1/3$, and the corresponding cone-volume is equal to

$$V\left(\frac{1}{3}\right) = \frac{32\,\pi\,R^3}{81}$$

Finally since, as it can be shown, the volume of the ball is equal to $4\,\pi\,R^3/3$, the ratio between the two is equal to

$$\left(4\,\pi\,R^3/3\right)\Big/V\left(\frac{1}{3}\right) = \frac{27}{8} \approx 3.375$$

2

One corner of a page of width $a = 8$ inches is folded over to just reach the opposite side as indicated in Figure 2. After expressing the length L of the crease in terms the angle θ, find the width x of the part folded over when L is a minimum.

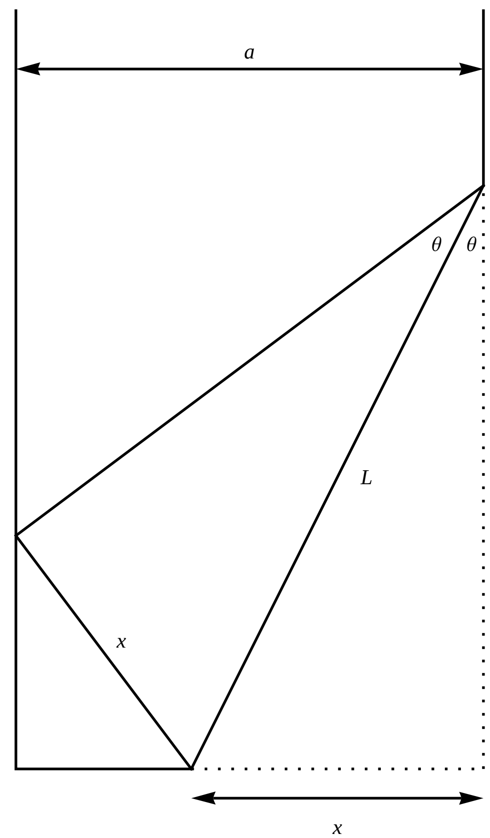

Figure 2.

2.1 solution

Just above the lower arrow head, the three angles reading from right to left are $\pi/2-\theta$, $\pi/2-\theta$, and $2\,\theta$. Thus, we have

$$\cos(2\,\theta)=\frac{a-x}{x},\; 2\cos(\theta)^2=1+\cos(2\,\theta)=\frac{a}{x},\; x=\frac{a}{2\cos(\theta)^2} \tag{2.1}$$

With $L=\frac{x}{\sin(\theta)}$, we obtain $L(\theta)=\frac{a}{2\sin(\theta)\cos(\theta)^2}$. Differentiation $L(\theta)$, we get

$$L'(\theta)=\frac{a}{2}\,\frac{3\sin(\theta)^2-1}{\sin(\theta)^2\cos(\theta)^3}$$

and therefore $\sin(\theta)^2=1/3$, and

$$x=\frac{a}{2\cos(\theta)^2}=\frac{a}{2\left(1-\sin(\theta)^2\right)}=\frac{a}{2\times\frac{2}{3}}=\frac{3\,a}{4}$$

So, if $a=8$, then $x=6$.

3

Suppose that $a_1,\,a_2,\,\ldots,\,a_n$ are real numbers such that the function

$$f(x)=a_1\sin(x)+a_2\sin(2\,x)+\ldots+a_n\sin(n\,x) \tag{3.1}$$

satisfies $|f(x)|\le|\sin(x)|$, for all real numbers x. Prove that

$$|a_1+2\,a_2+\ldots+n\,a_n|\le 1 \tag{3.2}$$

3.1 solution

Note that

$$|a_1+2\,a_2+\ldots+n\,a_n|=|f'(0)|=\left|\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}\right|=\left|\lim_{x\to 0}\frac{f(x)}{x}\right|=\lim_{x\to 0}\left|\frac{f(x)}{x}\right|\le \lim_{x\to 0}\left|\frac{\sin(x)}{x}\right|=1 \tag{3.3}$$

4

For what values of p does the series $\sum_{n=6}^{\infty}\left(e^{-\frac{1}{n^2}}+\frac{1}{n^2}-1\right)^p$ converge? Fully justify your answer.

4.1 solution

For small x we have $e^x \approx 1 + x + x^2/2$. Therefore, intuitivelly, for large n, $e^{-\frac{1}{n^2}} \approx 1 - \frac{1}{n^2} + \frac{1}{2n^4}$, meaning that $e^{-\frac{1}{n^2}} + \frac{1}{n^2} - 1 \approx \frac{1}{2n^4}$. Formally, we apply the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{\left(e^{-\frac{1}{n^2}} + \frac{1}{n^2} - 1\right)^p}{\frac{1}{n^{4p}}} = 2^{-p} \tag{4.1}$$

so that $\sum_{n=6}^{\infty} \left(e^{-\frac{1}{n^2}} + \frac{1}{n^2} - 1\right)^p$ converges iff $\sum_{n=6}^{\infty} \frac{1}{n^{4p}}$ does, which happens iff $4p > 1$, i.e., iff $p > \frac{1}{4}$.

5

Show that the limit $\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n)\right)$ exists, and find an upper and lower bound for γ.

5.1 solution

Set $\Gamma(n) := \sum_{k=1}^{n} \frac{1}{k} - \log(n)$, where n is a positive integer. Then

$$\Gamma(n+1) - \Gamma(n) = \frac{1}{n+1} + \log(n+1) - \log(n) = \frac{1}{n+1} + \int_n^{n+1} \frac{1}{x}\, dx < 0 \tag{5.1}$$

so that the sequence $\Gamma(n)$ is decreasing. Also,

$$\log(n+1) - \log(2) = \int_1^n \frac{1}{x+1}\, dx \le \sum_{k=2}^{n} \frac{1}{k} \tag{5.2}$$

and therefore

$$\log\left(\frac{n+1}{n}\right) - \log(2) + 1 \le \sum_{k=1}^{n} \frac{1}{k} - \log(n) \tag{5.3}$$

Therefore, since $\lim_{n\to\infty} \log\left(\frac{n+1}{n}\right) = 0$,

$$1 - \log(2) \le \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n)\right) = \gamma \le \Gamma(n) = \sum_{k=1}^{n} \frac{1}{k} - \log(n) \tag{5.4}$$

for any integer $n \ge 1$. For example, setting $n = 2$,

$$1 - \log(2) \le \gamma \le 1 - \log(2) + \frac{1}{2} \tag{5.5}$$

or $0.306853 \le \gamma \le 0.806853$ (the “exact” value being equal to $\gamma = 0.577216$ ...).

6

Calculate the following integral and check your result by differentiation.

$$\int (\ln x)^2 \, dx \tag{6.1}$$

6.1 solution

Setting $u = (\ln x)^2$, $dv = dx$; $du = 2\,(\ln x)\,\frac{1}{x}\,dx$, $v = x$, by integration by parts we get

$$\int (\ln x)^2 \, dx = x\,(\ln x)^2 - 2\int \ln x \, dx$$

Setting $u = \ln x$, $dv = dx$; $du = \frac{1}{x}\,dx$, $v = x$, by integration by parts we get

$$\int (\ln x)^2 \, dx = x\,(\ln x)^2 - 2\int \ln x \, dx = 2\,x + x\,(\ln x)^2 - 2\,x \ln x.$$

7

Calculate the following integral and check your result by differentiation.

$$\int \frac{1}{x^7 - x} \, dx \tag{7.1}$$

7.1 solution

Trying

$$\frac{1}{x^7 - x} = \frac{1}{x\left(x^6 - 1\right)} = \frac{A}{x} + \frac{B\,x^5}{x^6 - 1}$$

we gat $A = -1$, $B = 1$, so that (assuming $0 < x < 1$)

$$\int \frac{1}{x^7 - x} \, dx = -\int \frac{1}{x} \, dx - \int \frac{x^5}{1 - x^6} \, dx = \frac{1}{6} \ln\left(1 - x^6\right) - \ln(x)$$

Checking,

$$\left(\frac{1}{6} \ln\left(1 - x^6\right) - \ln(x)\right)' = -\frac{x^5}{1 - x^6} - \frac{1}{x} = \frac{1}{x^7 - x}.$$