U.C. MATH BOWL 2019

LEVEL I— Session 1

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Two lines with slopes m and n, with $m > n > 0$, intersect at the origin. The line $y = x$ bisects the angle between the two lines. What is the value of mn?

> Because the line $y = x$ is an angle bisector, the lines are symmetric with respect to reflection in this line. If (x, nx) is one point on the line of slope n, it's reflection (nx, x) is on the other line. But another expression for this point is $(nx, m(nx))$. The second coordinates must be the same since the first coordinates are. So $mn = 1$.

2. At a neighborhood potluck 3 Singhs, 3 Zhangs, and 2 Slavins sit at random around a circular table. What is the chance that each person is sitting next to at least one person from another family?

> Answer: 53/70. Number the seats and treat members of each family as indistinguishable. Then there are $560 = \frac{8!}{3!3!2!}$ ways for the 8 people to be seated.
>
> There are $80 = 8\frac{5!}{2!3!}$ ways that the Singhs can sit together so that one of them is not seated next to someone from another family. That is, put the middle Singh in any of 8 seats and fill in the other seats with the remaining families in $\frac{5!}{2!3!}$ ways. Of course, there are also 80 ways that the Zhangs can be seated together.
>
> Finally, there are $24 = 3 \cdot 8$ ways that both the Zhangs and the Singhs can sit together. That is, put the middle Zhang in any one of 8 seats and then there's just 3 choices for the middle Singh.
>
> Therefore there are $80 + 80 - 24 = 136$ ways $(|A| + |B| - |A \cap B|)$ that someone is surrounded by family members.
>
> The chance no one is surrounded by family members is then $\frac{560-136}{560}$.

3. Suppose a and b are positive numbers with the amazing property that

$$\log_a(b) - \log_b(a) = 3.$$

What is $(\log_a(b))^2 + (\log_b(a))^2$?

> Let $x = \log_a(b)$ so that $a^x = b$. From this we see that $b^{1/x} = a$ so that $\log_b(a) = 1/x$. We're told that
>
> $$x - \frac{1}{x} = 3,$$
>
> from which we learn, by squaring, that
>
> $$x^2 - 2 + \frac{1}{x^2} = 9.$$
>
> That means
>
> $$x^2 + \frac{1}{x^2} = 11.$$

4. Two numbers a and b are such that the equation $\sin x + a = b\,x$ has exactly two solutions. Prove that the system

$$\sin x + a = b\,x, \quad \cos x = b$$

has at least one solution. Hint: Draw a picture!

For the equation $\sin x + a = bx$ to have exactly two solutions, $x = x_1$ and $x = x_2$, the line $y = bx - a$ must be tangent to the curve $y = \sin x$ at one of the solution points, say at $x = x_1$ (otherwise, there will be either 1 or 3 solutions – draw a picture). That means that $\frac{d}{dx}(\sin x)|_{x=x_1} = \cos x_1 = b$. Thus, the second equation is automatically satisfied.

5. Suppose f is a continuously differentiable function, that $f'(2) = \pi$ and that $f(2) = -4$. Find the limit

$$\lim_{x \to 2} \frac{f(x) + 4}{\sqrt{x} - \sqrt{2}}.$$

For this limit, which is indeterminate with form $0/0$, we see the limit is

$$\lim_{x \to 2} \frac{f'(x)}{\frac{1}{2\sqrt{x}}} = \pi \cdot 2\sqrt{2}.$$

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LEVEL I — Session 2

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. The three-digit numbers $C99$, $A6A$, $BC7$ and $B91$ form an arithmetic sequence in this order.

   Capital letters represent digits so in this notation XYZ is the number $100X + 10Y + Z$.

   Recall that in an arithmetic sequence the difference between one term and the next is a constant. What is the value of $A2 + B2 + C2$?

> $A = 3, B = 4, C = 2$ so $2(A + B + C) = 18$. The constant difference for the progression is $B91 - BC7 = 84 - 10C$.
>
> Therefore $101A + 60 - 100C - 99 = 84 - 10C$ which we may write as
>
> $$101A = 90C + 123.$$
>
> We're in a Diophantine situation here — we want solutions in integers, so conclude (because A and C are digits) $A = 3$ and $C = 2$. (Technically, an analysis of the situation starts with nothing the gcd of 101 and 90 is 1; but inspection yields the solution easily.) Then any difference involving B shows $B = 4$.

2. What is the sum of all the integer values of x for which $|3x - 3| < 13$?

> Unfold the absolute value and write
>
> $$-13 < 3x - 3 < 13,$$
>
> $$-10 < 3x < 16,$$
>
> $$-10/3 < x < 16/3.$$
>
> Since x is an integer this says
>
> $$-3 \leq x \leq 5$$
>
> Adding the consecutive integers in this range is easy because of cancellation. The sum is 9.

3. If n is a positive integer write $s(n)$ for the sum of n's digits. So, for example, $s(543) = 5+4+3 = 12$.

   (a) What is $s(1) + s(2) + s(3) + \cdots + s(9)$?

   (b) What is $s(1) + s(2) + s(3) + \cdots + s(10^4)$? (hint: how does part (a) help here?)

> Answers: 45 and a big number described below.
>
> Start by noting or calculating that $s(0) + \cdots + s(9) = (1/2) \cdot 9 \cdot 10 = 45$.

For adding up the digit sums from 0 to 99, you could organize the work this way:

0	10	20	$\cdots$	90
1	11	21	$\cdots$	91
2	12	22	$\cdots$	92
3	13	23	$\cdots$	93
4	14	24	$\cdots$	94
5	15	25	$\cdots$	95
6	16	26	$\cdots$	96
7	17	27	$\cdots$	97
8	18	28	$\cdots$	98
9	19	29	$\cdots$	99

The sum of 0 to 9 occurs in each of the 10 columns as the right hand digits of the numbers. And, each of $0, 1, 2, \cdots 9$ occurs 10 times as the left digit in one of the columns. That says

$$s(0) + \cdots + s(99) = 10(45) + 10(45) = 900.$$

Adding in $s(0)$ adds nothing; including $s(100)$ in the sum adds 1.

So, in summing the digits of the integers 0 through 99 we see that each digit occurs 10 times in in the right hand place of a number (once each decade) and 10 times as a left-most digit. So the sum $s(0) + \cdots + s(99) = 20 \times 45 = 900$. Add one more to include $s(100)$ in the sum and you'll get 901.

In summing the digits of the numbers from 0 to 999 we can organize the calculation in a similar way and find that the the sum just computed occurs once for each of 10 left-hand digits while each of the left-hand digits appears 100 times.

So the sum of the digits of the integers from 0 to 999 is $100(45) + 10(900) = 13500$.

Add one more for the sum of the digits of 1000 to get a total sum of 13501.

So: in summing the digits of the integers 0 through 99 we see that each digit occurs 10 times in in the right hand place of a number (once each decade) and 10 times as a left-most digit. So the sum $s(0) + \cdots + s(99) = 20 \times 45 = 900$. Add one more to include $s(100)$ in the sum and you'll get 901.

The next number in this pattern, $s(0) + \cdots + s(10000)$, is $1000(45) + 10(13500) + 1$ since the sum just calculated occurs once for each of 10 left hand digits and Each of the digits appears as the left digit of 1000 numbers. We remember to add 1 for $s(10^4)$.

4. If

$$\sin(x) + \sin^2(x) + \sin^3(x) + \cdots = 4$$

what could be the value of

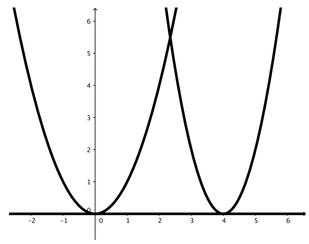
$$\cos(x) + \cos^2(x) + \cos^3(x) + \cdots ?$$

$$\sin(x) + \sin^2(x) + \sin^3(x) + \cdots = \frac{\sin x}{1 - \sin x}$$

so $\sin x = 4/5$. That says

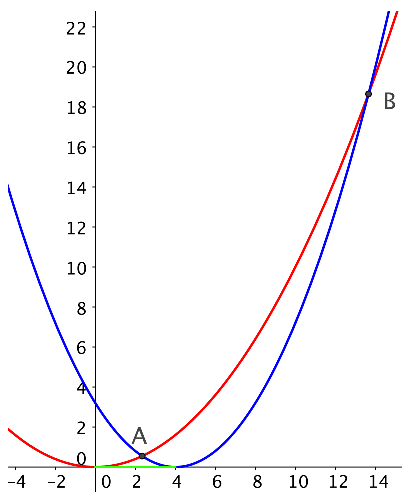
$$\cos x = \sqrt{1 - 16/25} \quad \text{or} \quad \cos x = -\sqrt{1 - 16/25}.$$

5. Find the area of the region bounded by the graphs of $y = x^2$, $y = 2(x-4)^2$, and $y = 0$.



The picture accompanying the question doesn't tell the whole story, does it? After all, the two parabolas meet at *two* points.

This figure illustrates the area bounded by the three curves. The functions were divided by 10 before plotting in order to better illustrate the situation.



You can find A and B by solving a quadratic equation for their x-coordinates. If a, b are the x-coordinates of the points A, B where the parabolas meet, then the area of the bounded region is

$$A = \int_0^4 \text{red}\, dx + \int_4^b \text{red}\, dx - \int_4^b \text{blue}\, dx$$

or, doing some algebra first to minimize the calculus required,

$$A = \int_0^4 \text{red}\, dx + \int_4^b \text{red} - \text{blue}\, dx$$