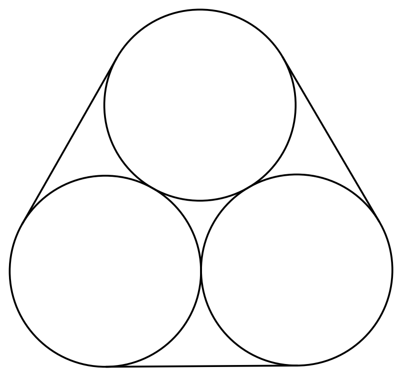
U.C. MATH BOWL 2019

LEVEL II — Session 1

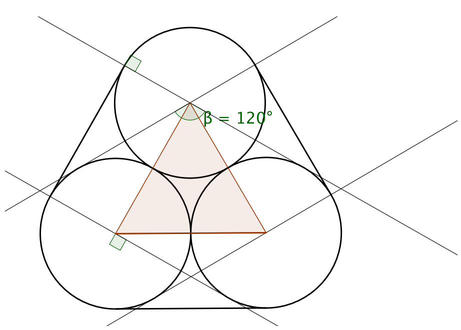
Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. The figure shows three pipes, in cross-section, bound together by a strong band. If the pipes are each 1 foot in diameter how long is the band?



Three feet plus the circumference of one of the pipes. The straight stretches of the band are parallel to and congruent to the line segments connecting the centers of the circles since tangency at a point implies they're perpendicular to radii to that point.



2. How many three digit integers (i.e. from 100 to 999) have the property that none of their adjacent digits are the same?

If the middle digit is 0 then the left and right digits can each be selected in 9 ways, for a total of 81 such integers. If the middle digit is any of the 9 nonzero digits, the left digit can be selected in 8 ways (not zero and not equal to the middle digit) and the right digit can be selected in 9 ways, for a total of $9 \times 8 \times 9$ integers. In all, that gives $81 + 81(8) = 729$.

3. If n is a positive integer write $s(n)$ for the sum of n's digits. So, for example, $s(543) = 5+4+3 = 12$.

(a) What is $s(1) + s(2) + \cdots + s(9)$?

(b) What is $s(1) + s(2) + s(3) + \cdots + s(1000)$? (hint: how does part (a) help here?)

Answers: 45, 13501.

Start by noting or calculating that $s(0) + \cdots + s(9) = 45$.

For adding up the digit sums from 0 to 99, you could organize the work this way:

0	10	20	$\cdots$	90
1	11	21	$\cdots$	91
2	12	22	$\cdots$	92
3	13	23	$\cdots$	93
4	14	24	$\cdots$	94
5	15	25	$\cdots$	95
6	16	26	$\cdots$	96
7	17	27	$\cdots$	97
8	18	28	$\cdots$	98
9	19	29	$\cdots$	99

The sum of 0 to 9 occurs in each of the 10 columns as the right hand digits of the numbers. And, each of $0, 1, 2, \cdots 9$ occurs 10 times as the left digit in one of the columns. That says

$$s(0) + \cdots + s(99) = 10(45) + 10(45) = 900.$$

Adding in $s(0)$ adds nothing; including $s(100)$ in the sum adds 1.

So, in summing the digits of the integers 0 through 99 we see that each digit occurs 10 times in in the right hand place of a number (once each decade) and 10 times as a left-most digit. So the sum $s(0) + \cdots + s(99) = 20 \times 45 = 900$. Add one more to include $s(100)$ in the sum and you'll get 901.

In summing the digits of the numbers from 0 to 999 we can organize the calculation in a similar way and find that the the sum just computed occurs once for each of 10 left-hand digits while each of the left-hand digits appears 100 times.

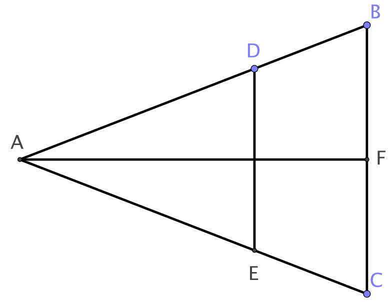
So the sum of the digits of the integers from 0 to 999 is $100(45) + 10(900) = 13500$.

Add one more for the sum of the digits of 1000 to get a total sum of 13501.

4. A and B are positive, two-digit numbers so that $AB = 9009$. What is $A + B$?

Since $AB = 9009 = 9(11)(91)$ the only product or two two-digit integers equal to 9009 is $99 \cdot 91$. So $A + B = 190$.

5. Consult the figure. Points D and E are situated so that both $\triangle ABC$ and $\triangle ADE$ are isosceles with $AB = AC$ and $DE \parallel CB$. F is the midpoint of BC.



Suppose that the area and the perimeter of $\triangle ADE$ and trapezoid $DBCE$ are equal.

Calculate $|FB|/|AB|$.

Answer $\sqrt{2}-1$.

Since this is a question about ratios of lengths, there's no problem with assuming a specific value for a particular length. We assume that $|AD| = 1$.

Because the trapezoid and smaller triangle have the same area, the ratio of the larger to the smaller triangle is $2:1$ and that tells us that the ratios of their sides is $\sqrt{2}:1$. So, for example, $|DB| = \sqrt{2}-1$.

The equality of perimeters equation tells us that $|FB|/|AB| = \sqrt{2}-1$.

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LEVEL II — Session 2

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Three people stand in a line facing the same direction. Person C is at the end of line and can see A and B. Person B is in the middle and can only see A. And, A can see no one.

   A puzzle master shows the people that she has 3 blue hats and 2 red hats. With their eyes closed, she puts one of these hats on each person's head and then invites them to open their eyes.

   Person C says, "I can't tell what color hat is on my head."

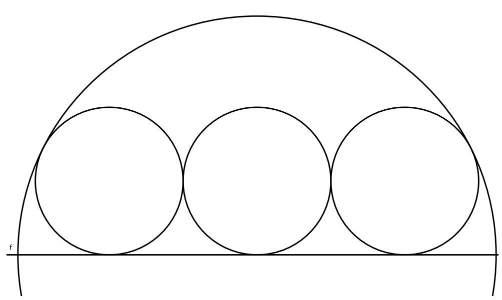
   Person B says, "I can't tell what color hat is on my head."

   Person A says, "I know what color hat is on my head."

   How does person A know, and what color is the hat?

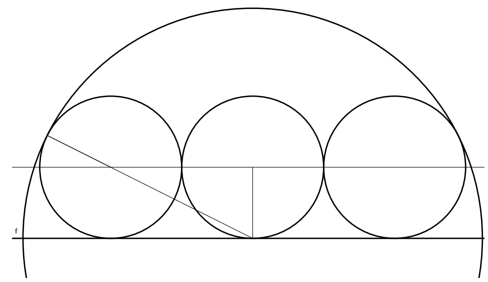
C can't be looking at two red hats or he would know his hat is blue. B knows this and, if looking at a red hat, would know her hat was not also red, and so would know it is blue. So B can see a blue hat on A's head. A knows this and so knows she's wearing a blue hat.

2. The centers of three circles of radius 1 lie on a line and the outer two circles are tangent to the one in the middle as shown in the figure. A semicircle is circumscribed around these smaller circles so that its diameter is tangent to all three of them and so that it is tangent to the outer two small circles. What is the radius of the semicircle?



Answer: $\sqrt{5}+1$.

This picture indicates how to use Pythagoras to determine the radius of the semicircle. The radius connecting the center of the semicircle to the point of tangency with a smaller circle must pass through the center of the smaller circle.



3. Suppose that $f(n)$ is defined by

$$f(n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n^2-1 & \text{if } n \text{ is odd} \end{cases}$$

Find all the numbers n so that $f(f(n)) = 8$.

If n is even, $f(n)$ is odd and so $f(f(n)) = (n-1)^2-1$. If n is odd, $f(n)$ is even and $f(f(n)) = n^2-2$. So

$$f(f(n)) = \begin{cases} n^2-2n & n \text{ even} \\ n^2-2 & n \end{cases}$$

$n^2-2n-8=0$ says $n = (2 \pm \sqrt{36})/2$ so $n = 4, -2$

4. In a certain country, every 20th mathematician is a writer, while every 60th writer is a mathematician. Are there more writers or mathematicians in the country? How many times more?

Let M be the number of mathematicians and W the number of writers in the country. Consider the set of those that are both. The size of this set is $M/20$ and also $W/60$. Thus,

$$\frac{M}{20} = \frac{W}{60} \qquad \Longrightarrow \qquad W = 3M.$$

Thus, there are three times as many writers as there are mathematicians.

5. How many non-congruent triangles with sides of integer lengths and with perimeter 7 are there?

There are 2 such non-congruent triangles.

If the sides are a, b, c then $a+b > c$ so $7 = a+b+c > 2c$. Hence $c < 7/2$. Nothing special about c so the only possible side lengths are $1, 2, 3$.

We may assume $a \leq b$ and enumerate the possible triangles:

a	b	c
1	1	5X
1	2	4X
1	3	3
2	2	3
2	3	2
3	3	1

So triangles with sides $1, 3, 3$ and $2, 2, 3$ each of which appears twice in this list are the only possibilities.