# U.C. MATH BOWL 2019

LEVEL III — Session 1

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Peter was 10 the day before yesterday. Next year, he'll be 13. Explain how this is possible.

> If yesterday was Pete's birthday, he turned 11 then. If it is now new year's day then he'll be 12 this year, and 13 the year after that.

2. A train that is 180 meters long passes a signal in 90 seconds. How long will this train take to pass over a bridge that is 360m long?

> Each car on the train is moving at 2 m/s. The first car in the train crosses the bridge in 180 seconds. At that point, the last car is 180 m from the end of the bridge and it takes that car 90 seconds to get there. So the total time to cross the bridge is 270 seconds.

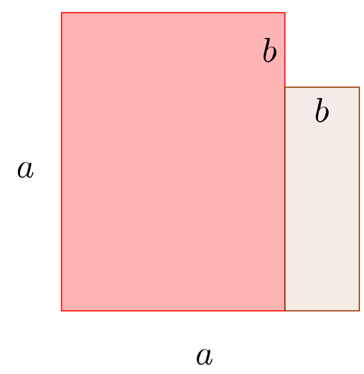
3. If Sean can eat a cake in 30 minutes by himself; Barbara, in 1 hour; and Michael, in 10 minutes, how long will it take for them to eat a cake together?

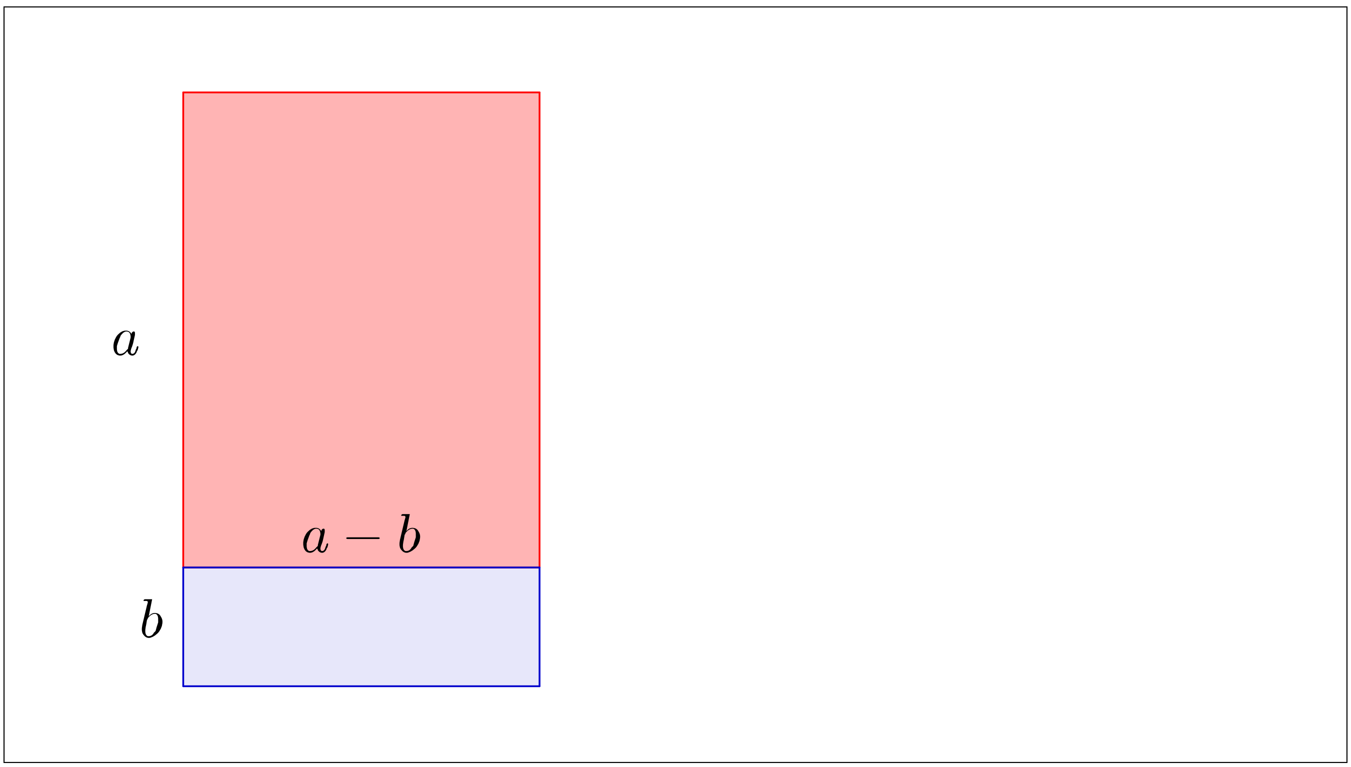
> In one hour Sean eats 2 cakes, Barbara eats 1 , and Mike eats 6. So, together, they can eat 9 cakes in an hour. That's the same as 1 cake in 1/9 hour.
>
> Some people like to think about this a little differently. They say Sean eats cake at the rate of 1/30 cake per minute, Barbara at 1/60 cake per minute, and Mike at 1/10 cake per minute. Working together they eat cake at the rate
>
> $$\frac{1}{30} + \frac{1}{60} + \frac{1}{10} \text{cakes per minute.}$$
>
> So together they take $M$ minutes to eat one cake where
>
> $$\left(\frac{1}{30} + \frac{1}{60} + \frac{1}{10}\right) M = 1.$$
>
> Personally, I like the first approach much better because I can do it in my head.

4. Jackie sold two cars for \$25,000 each. The first car sold for a profit of 22%, and the second sold at a loss of 7%. What was the total percent profit on the sale of the two cars? Express your answer to the nearest hundredth.

> If the first car was worth $x$ dollars, we're told that $25000 = (1.22)x$. And if the second car was worth $y$ dollars, $25000 = (0.93)y$.
>
> So
>
> $$x + y = 25000(\frac{1}{1.22} + \frac{1}{0.93}).$$

5. Show how to rearrange the two shaded rectangles in the figure to prove that $a^2 - b^2 = (a+b)(a-b)$. Explain!



# U.C. MATH BOWL 2019

LEVEL III — Session 2

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Sally as a bunch of quarters and nickels in her pocket worth \$3.35. If the quarters are switched into dimes and the nickels into pennies, she'll have \$1.27. How many quarters and nickels does she have?

> 12 and 7. If $Q$ and $N$ are the number of quarters and nickels that Sally has, we're told that
>
> $$Q(25) + N(5) = 335$$
> $$Q(10) + N(1) = 127$$
>
> There are ways to solve these two equations, but note that Sally can have at most 13 quarters since $25Q \leq 335$. So there aren't many possibilities to check.

2. Some questions about fractions!

   (a) What number is greater, $\frac{2017}{2018}$ or $\frac{2018}{2019}$? Why?

> You *could* just use a calculator and compare decimal representations of these numbers. But sometimes we're better off NOT using a calculator to answer even an easy question because we learn, or remember, important ideas when we solve problems in other ways. Of course, a calculator won't help much with addressing the second part of the question, so trying additional approaches to even an easier question could give us valuable experience.
>
> Why not benchmark the two fractions against 1? To do this, consider the fractions
>
> $$\frac{2018}{2018} - \frac{2017}{2018} \quad \text{and} \quad \frac{2019}{2019} - \frac{2018}{2019}.$$
>
> The first is 1/2018 and the second is 1/2019 so the second is smaller by a "same number of different sized parts" comparison. Clearly $\frac{2018}{2019}$ is closer to 1 than $\frac{2017}{2018}$ and so it is larger.

   (b) If you have three years in a row, Year 1, Year 2, Year 3, which is bigger

$$\frac{\text{Year 1}}{\text{Year 2}} \quad \text{or} \quad \frac{\text{Year 2}}{\text{Year 3}}?$$

   Please explain. (Hint let Year 1 be $k$...)

> The same approach should work! The idea of doing a benchmark against 1 serves just as well here as in the previous question:
>
> $$\frac{\text{Year 2}}{\text{Year 2}} - \frac{\text{Year 1}}{\text{Year 2}} = \frac{1}{\text{Year 2}}$$
> $$\frac{\text{Year 3}}{\text{Year 3}} - \frac{\text{Year 2}}{\text{Year 3}} = \frac{1}{\text{Year 3}}$$

The second difference is smaller (same number, different sized parts comparison) and so $\frac{\text{Year 2}}{\text{Year 3}}$ is larger as it is closer to 1 than $\frac{\text{Year 1}}{\text{Year 2}}$.
Alternatively, you could "cross multiply" saying that

$$\frac{k}{k+1} < \frac{k+1}{k+2}$$

exactly when

$$k(k+2) < (k+1)^2.$$

But $(k)(k+2) = k^2 + 2k$ while $(k+1)^2 = k^2 + 2k + 1$. The second of these is always larger. Of course this algebra isn't necessary at all for answering the question!

3. If $n$ is a positive integer write $s(n)$ for the sum of $n$'s digits. So, for example, $s(543) = 5+4+3 = 12$.

(a) $s(1) + s(2) + \cdots + s(9) = ?$

(b) $s(1) + s(2) + s(3) + \cdots + s(100) = ?$ (hint: how does part (a) help here?)

Answers: 45 and 901.

Start by noting or calculating that $s(0) + \cdots + s(9) = 45$.

For adding up the digit sums from 0 to 99, organize your thinking this way:

| | | | | |
|---|---|---|---|---|
| 0 | 10 | 20 | $\cdots$ | 90 |
| 1 | 11 | 21 | $\cdots$ | 91 |
| 2 | 12 | 22 | $\cdots$ | 92 |
| 3 | 13 | 23 | $\cdots$ | 93 |
| 4 | 14 | 24 | $\cdots$ | 94 |
| 5 | 15 | 25 | $\cdots$ | 95 |
| 6 | 16 | 26 | $\cdots$ | 96 |
| 7 | 17 | 27 | $\cdots$ | 97 |
| 8 | 18 | 28 | $\cdots$ | 98 |
| 9 | 19 | 29 | $\cdots$ | 99 |

The sum of 0 to 9 occurs in each of the 10 columns as the right hand digits of the numbers. And, each of $0, 1, 2, \cdots 9$ occur 10 times as the left digit in one of the columns. That says

$$s(0) + \cdots + s(99) = 10(45) + 10(45) = 900.$$

Adding in $s(0)$ adds nothing; including $s(100)$ in the sum adds 1.

In summing the digits of the integers 0 through 99 we see that each digit occurs 10 times in the right hand place of a number (once each decade) and 10 times as a left-most digit. So the sum $s(0)+\cdots+s(99) = 20 \times 45 = 900$. Add one more to include $s(100)$ in the sum and you'll get 901.

4. Using the integers 1 through 9, each exactly once, fill a $3 \times 3$ grid so that the product of the numbers in the first row is 12, the product of the numbers in the second row is 112, the product of the numbers in the first column is 216, and the product of the numbers in the second column is 12.

| | | | |
|---|---|---|---|
| | | | 12 |
| | | | 112 |
| | | | |
| 216 | 12 | | |

The number 5 can't appear in the first or second row or first or second column since the products of these are not divisible by 5. So 5 must be in the lower right box.

We can obtain a product to 12 in just two ways ($3 \times 4$ and $2 \times 6$) so those pairs must appear in the first row and second column. Therefore 1 must appear in the middle box of the first row.

Three factors of 3 must appear in the first column. If 6 contributes one of these 3's then 6 is in the upper left corner. The other entries in the column would then be 9 and 4. But 4 would need to occur in the middle column if 6 is in the first row. Consequently, 3 and 9 appear in the first column and so the remaining number there is 8.

The first row is then $3, 1, 4$.

Since 9 doesn't divide 112, it must be that 9 is in the bottom left corner and the 8 is in the middle box of the first column.

7 must appear in the middle row and 6 can not.

| | | | |
|---|---|---|---|
| 3 | 1 | 4 | 12 |
| 8 | 2 | 7 | 112 |
| 9 | 6 | 5 | |
| 216 | 12 | | |

5. Three people stand in a line facing the same direction. Person C is at the end of line and can see A and B. Person B is in the middle and can only see A. And, A can see no one.

   A puzzle master shows the people that she has 3 blue hats and 2 red hats. With their eyes closed, she puts one of these hats on each person's head and then invites them to open their eyes.

   Person C says, "I can't tell what color hat is on my head."

   Person B says, "I can't tell what color hat is on my head."

   Person A says, "I know what color hat is on my head."

How does person A know, and what color is the hat?

> C can't be looking at two red hats or he would know his hat is blue. B knows this and, if looking at a red hat, would know her hat was not also red, and so would know it is blue. So B can see a blue hat on A's head. A knows this and so knows she's wearing a blue hat.