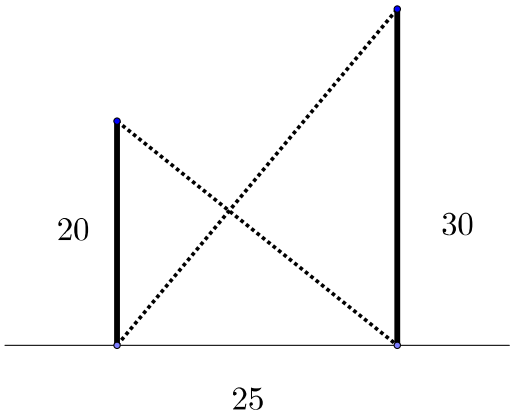
U.C. MATH BOWL 2018

LEVEL II — Session 1

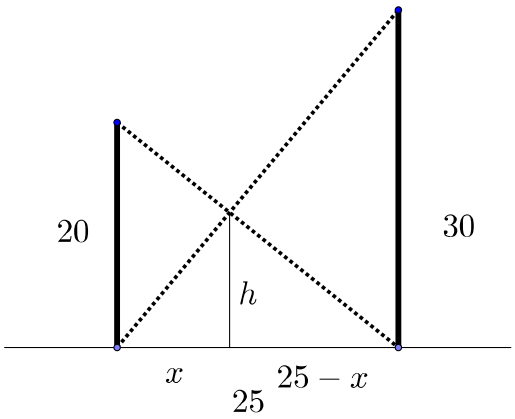
Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Two vertical poles with heights 20 and 30 feet are installed with their bases 25 feet apart. Chains are connected from the top of each pole to the base of the other pole. How high above the ground do the chains cross?



Answer: 12 feet. Introduce names for the vertical and horizontal location of where the chains cross relative to the base of one of the poles.



Similar triangles tell us $h : x = 30 : 25$ and$h : 25 - x = 20 : 25$. Written as ratios these proportions tell us $h/x = 30/25$ and $h/(25 - x) = 20/25$ which we write as

$$h = (30/25)x$$
$$h = (20/25)(25 - x)$$

There's lots of ways to solve these equations but perhaps the easiest is to use the first equation to eliminate the variable h in the second equation, showing that

$$(6/5)x = (4/5)(25 - x).$$

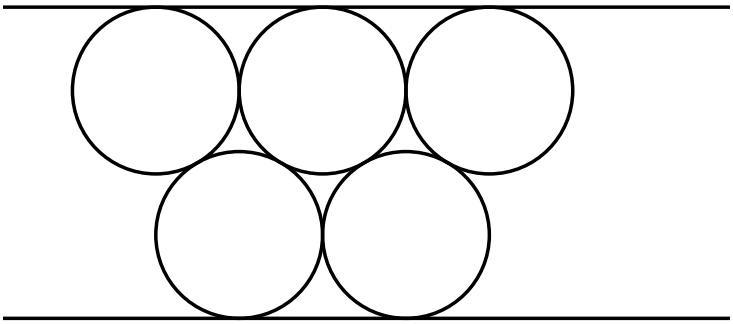
This amounts to saying $2x = 20$ so we see $x = 10$. Using this in either of the original equations, we learn that $h = 12$.

2. How many times in a 12 hour period do the hour and minute hands of a clock form a right angle?

Answer: 22. The minute hand moves 360 degrees in 60 minutes, or 6 degrees per minute. The hour hand moves 360 degrees in 12 hours. That is (360)/(12) degrees per hour which amounts to 360/720 = 1/2 degrees per minute. Since the minute hand travels 5.5 degrees per minute faster than the hour hand, it gains 90 degrees every 90/5.5 minutes.

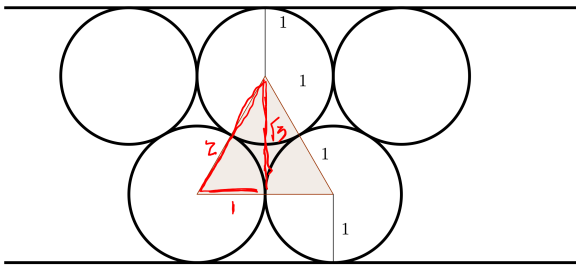
In 12 hours (720 minutes) the minute hand gains 90 degrees over the hour hand 720/(90/5.5) = 44 times. At every other of these times, the angular difference between the hands is 180 or 0 degrees. At the other 22 times, the hands are 90 degrees apart.

3. In the figure circles with diameter 2 are packed together in the pattern shown so that they just touch each other.



How far apart are the parallel lines that just touch the circles?

Answer: $2 + \sqrt{3}$.



Connect the centers of 3 mutually touching circles to obtain an equilateral triangle with side length 2. The altitude of this triangle is $\sqrt{2^2 - 1^2} = \sqrt{3}$.

4. These statements were all made on the same day:

   (a) Yesterday was Monday
   (b) Tomorrow is Friday
   (c) The day after tomorrow will be Friday
   (d) Tomorrow will be Saturday

(e) The day before yesterday was Tuesday

An additional interesting fact is this: *If you knew how many of the statements were correct, you could determine on which day of the week the statements were made.*

On what day of the week were the statements made?

Answer: Thursday. The statements amount to saying "Today is..."

(a) Tuesday
(b) Thursday
(c) Wednesday
(d) Friday
(e) Thursday

Consider, for each day of the week, how many of these statements are correct:

Day	Mon	Tue	Wed	Thur	Fri	Sat	Sun
# correct	0	1	1	2	1	0	0

If we knew 0 of the statements were correct we couldn't tell if the day was Mon, Sat, or Sun. And if we knew that exactly 1 of the statements was correct, the day might have been Tues, Wed, or Fri. In these cases, knowing the number of correct statements doesn't let us figure out the day. But if 2 of the statements are correct, we can tell that the day was a Thursday.

Since we're told we can determine the day from the number of correct statements, the day must be Thursday.

5. What is the last (right most) digit of the sum $2^{2018} + 3^{2018} + 7^{2018}$?

Answer: 2. We only need the remainder when this number is divided by 10.

Calculating modulo 10, note that $2^4 = 6$ and $6^k = 1$ so $2^{4k} = 6$. This says that

$$\begin{aligned} 2^{2018} &= (2^4)^{504}2^2 \\ &= 6^{504}2^2 \\ &= 2^4 2^2 \\ &= 4. \end{aligned}$$

Again working modulo 10, we have $3^4 = 7^4 = 1$. And so

$$\begin{aligned} 3^{2018} &= (3^4)^{504}3^2 \\ &= 1^{504}9 \\ &= 9 \end{aligned}$$

The same goes for 7. Hence, modulo 10,

$$2^{2018} + 3^{2018} + 7^{2018} = 4 + 9 + 9 = 2.$$