U.C. MATH BOWL 2018

LEVEL III — Session 1

Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. Suppose you use each of the digits $\{2, 3, 4, 6, 7, 8\}$ exactly once to write two three-digit numbers S and T so that $S - T$ is positive and as small as possible. What is $S - T$?

> Answer: 39. The hundreds places of S and T must be different since we can use each digit only once. So to make S and$T$ as close as possible we want the tens digit of S to be as small as possible (so 2) and the tens digit of T to be as big as possible (so 8). The hundreds digits of the numbers must be as close as possible (with a difference of just 1) and so they should be 4 and 3 or 7 and 6. So we conclude that $S = 42x$ and $T = 38y$ or $S = 72x$ and $T = 68y$ where in these two possible situations the x and y stand for some assignment of the remaining, unused digits. As before the larger of these digits should be assigned to T and the smaller of them to S in order to minimized the difference $S - T$. So $S = 426$ and $T = 387$ or $S = 723$ and $T = 684$. In either case the difference is 39.

2. On a card there are three statements listed:

   1 This list has exactly one false statement.

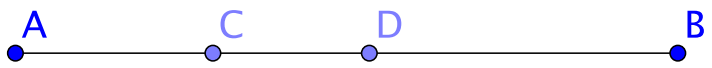
   2 This list has exactly two false statements.

   3 This list has exactly three false statements.

   How many true statements are there on the card?

> Answer: 1. Here, the statements are mutually contradictory so either 1 or 0 of them are true. If 1 is true, the other two are false, making statement #2 the one true statement. It cannot be the case that 0 of the statements are true because if that were the case, statement #3 would be true.

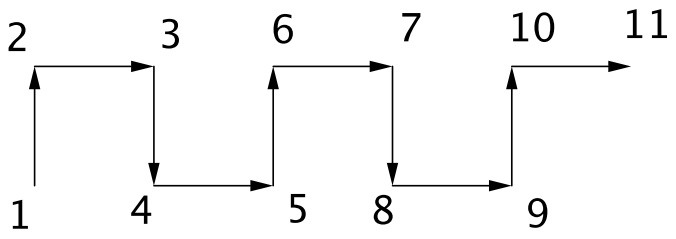
3. The segment AB with length 10 is divided by points C and D arranged as in the figure.



   The points C and D are placed in such a way that $2(|AC| + |DB|) = 3|CD|$. How long is CD?

> Answer: 4 units. Since $|AC| + |CD| + |DB| = 10$ we can write the condition as $2(10 - |CD|) = 3|CD|$ so that $20 = 5|CD|$ and so $|CD| = 4$.
>
> A slight generalization worth contemplating is one in which we let AB have arbitrary length L. Then $|CD| = (2/5)L$.

4. If the pattern shown in the picture is continued, in what direction does the arrow from 1000 to 1001 point?



Answer: to the right. Numbers are at the end of downward pointing arrows exactly when they are evenly divisible by 4, as is 1000. The next arrow in the pattern after a down arrow is a right arrow; that's what points from 1000 to 1001.

5. Tyler and Jesse each have a collection of candy. Tyler said to Jesse, "If you give me one piece of candy then we'll have the same number." Jesse replied, "but if you give me one piece then I'll have twice as many as you." How many pieces of candy do Tyler and Jesse have?

Answer: Tyler has 5, Jesse has 7. Let T represent the number of candies Tyler has and J represent the number that Jesse has. We're told $T + 1 = J - 1$ and $2(T - 1) = J + 1$.

There's a variety of ways to solve these two equations, including elimination of a variable. In this case, though, the easiest seems to be just adding these two equations which yields $T - 3 = 2$, showing that $T = 5$. With that, substitution into either of the original equations shows that $J = 7$.

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LEVEL III — Session 2

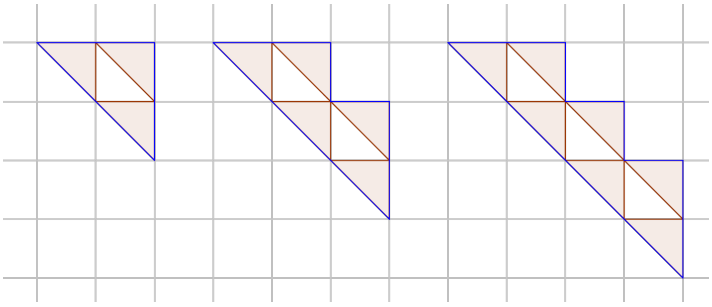
Instructions: Write your answers in the blue book provided. Remember that even correct answers without explanation may not receive much credit and that partially correct answers that show careful thinking and are well explained may receive many points.

Have Fun!

1. How many 3 digit numbers have the property that the sum of their digits is 24?

> 999 is too big by 3. Pick one digit to reduce by 3 in 3 ways. Pick one digit to reduce by 2 and one digit to reduce by 1 in 3×2 ways, or reduce each digit by 1. That's a total of 10 ways to do it. The numbers are $699, 969, 996, 789, 798, 978, 879, 987, 897$ and 888.

2. The picture below shows the first three figures in a growing triangle pattern. The grid squares in the figure have side length 1 and so their diagonals have length $\sqrt{2}$.



If the pattern continues growing in the same way, what will be the outer perimeter (shown in a different shade) of the 10th figure? What will be the outer perimeter of the Nth figure?

> Each time another triangle is added to the pattern one segment of length $\sqrt{2}$ and two segments of length 1 are added. The first pattern has perimeter $4 + 2\sqrt{2}$. Therefore the nth pattern will have perimeter
>
> $$4 + 2\sqrt{2} + (n-1)(2+\sqrt{2}).$$
>
> When $n = 10$ this is $22 + 11\sqrt{2}$.
>
> If one mistakenly assumes that the small triangles are equilateral with length 1 (or simply counts the number of sides of the smaller triangles that make up the perimeter of the pattern one gets the answers 33; $6 + 3(N-1)$. The outer perimeters of the first three figures are 6, 9, and 12. Each time a new triangle is added to each row, the outer perimeter increases by 3. The 10th figure will have outer perimeter $6 + 3(9) = 33$; the Nth figure will have an outer perimeter of $6 + 3(N1)$.

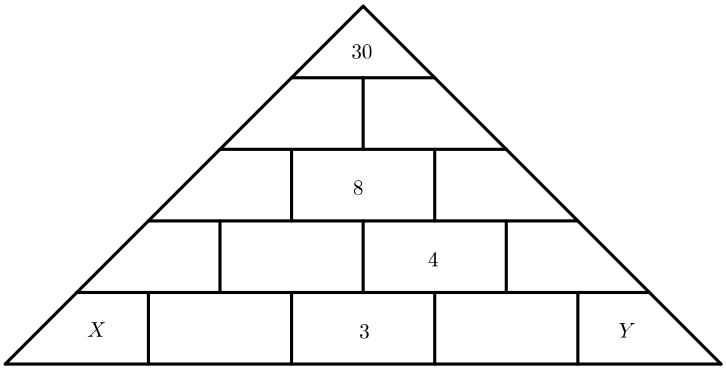
3. Challoney cleared out her inventory of fewer than 500 chocolate bars by putting them on sale for some whole number of cents less than their original price of 50 cents each. She sold everything for $31.93. What was the new price?

> Answer: 31 cents. If P and N are the new price in cents and the number of bars sold, we're told $PN = 3193$, $P < 50$, and $N < 500$. But $3193 = 31 \times 103$ is the only way to factor 3193. So $P = 31$ and $N = 103$.

4. A $3 \times 3 \times 3$ inch wooden cube is painted blue on four of its faces and is then cut into 27 unit cubes. Suppose you pick one of these cubes at random and roll it like a die. What is the chance that a blue face shows up on this roll?

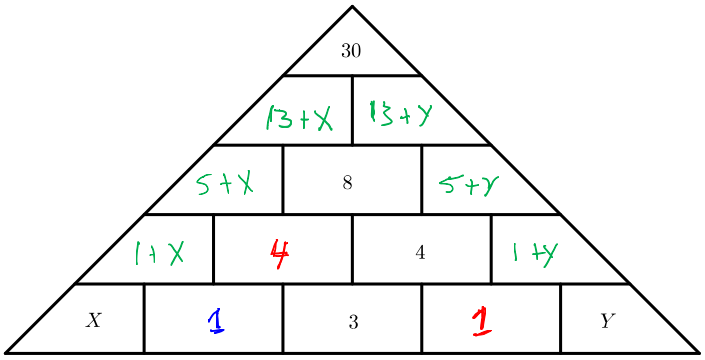
Answer: $36/(6 \times 27)$. There are $4 * (3 * 3) = 36$ painted faces on the unit cubes and 6(27) faces in all, so the chance of ending up with a blue face after the random selection and die roll procedure is just the chance of picking one of the 36 blue faces from a collection of 6(27 equally likely faces.

5. In the triangle shown the boxes are to be filled in with numbers so that the number in each box is the sum of the numbers in the two boxes below it.



What can you say about X and Y?

We can't say anything about X and Y individually but we can say something about their relationship: $X + Y = 4$.



We can fill in some of the boxes (in red) immediately based on the rule we're given, and then (in blue) one more box's number can be found. Working from bottom to top we fill in the green numbers from which we conclude that $26 + X + Y = 30$ so $X + Y = 4$.